

Homework 6: The ZX-Calculus

Quantum Information Systems Wesleyan University

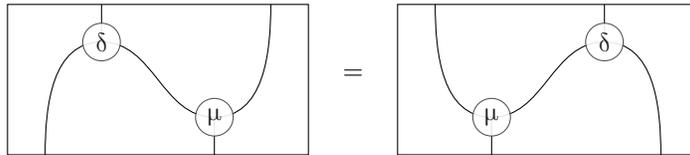
due 2017.05.08

Problem 1 (the Frobenius law)

Recall from lecture that the **Frobenius law** states:

$$(\delta \otimes \text{Id}) \cdot (\text{Id} \otimes \mu) = (\text{Id} \otimes \delta) \cdot (\mu \otimes \text{Id})$$

or diagrammatically,



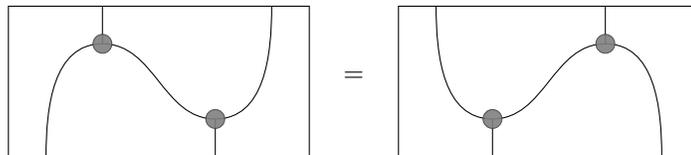
Recall also that the Hilbert space interpretations of Z- and X- basis spiders are given by:

$$\begin{aligned} \llbracket Z_m^n(\varphi) \rrbracket &:= |0^m\rangle\langle 0^n| + e^{i\varphi} |1^m\rangle\langle 1^n| \\ \llbracket X_m^n(\varphi) \rrbracket &:= |+\!^m\rangle\langle +\!^n| + e^{i\varphi} |-\!^m\rangle\langle -\!^n| \end{aligned}$$

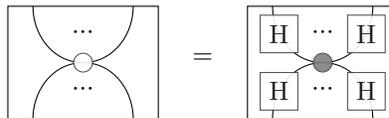
- a. Write the matrix representation for the spiders Z_2^1 and Z_1^2 .
- b. Use matrix arithmetic to show that Z-basis spiders with phase 0 satisfy the *Frobenius law*:

$$\llbracket (Z_2^1 \otimes \text{Id}) \cdot (\text{Id} \otimes Z_1^2) \rrbracket = \llbracket (\text{Id} \otimes Z_2^1) \cdot (Z_1^2 \otimes \text{Id}) \rrbracket$$

or diagrammatically:



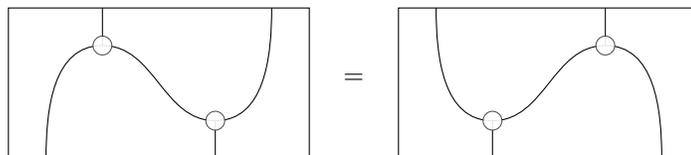
- c. Use the equation in part (b) together with the **color-change law**:



to show by diagram rewriting that X-basis spiders with phase 0 satisfy the Frobenius law:

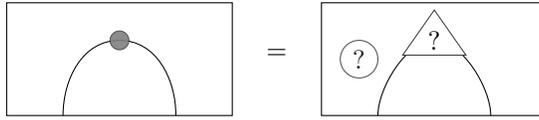
$$\llbracket (X_2^1 \otimes \text{Id}) \cdot (\text{Id} \otimes X_1^2) \rrbracket = \llbracket (\text{Id} \otimes X_2^1) \cdot (X_1^2 \otimes \text{Id}) \rrbracket$$

or diagrammatically:



Problem 2 (Frobenius implies zigzag)

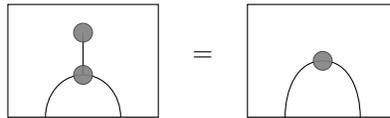
- a. Write an outer product representation for the interpretation of the spider Z_2^0 .
Up to a scalar, which familiar state is this? Record this result in the form of a diagrammatic equation by filling in the missing right-hand side below:



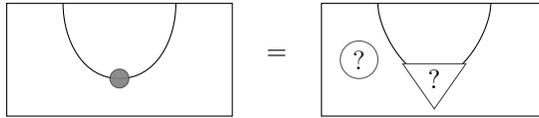
- b. Using either matrix arithmetic or algebra (your choice) verify that

$$[[Z_1^0 \cdot Z_2^1]] = [[Z_2^0]]$$

i.e.:



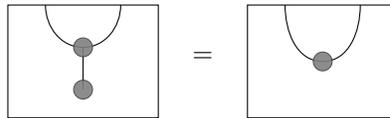
- c. Write an outer product representation for the interpretation of the spider Z_0^2 .
Up to a scalar, which familiar effect is this? Record this result in the form of a diagrammatic equation by filling in the missing right-hand side below:



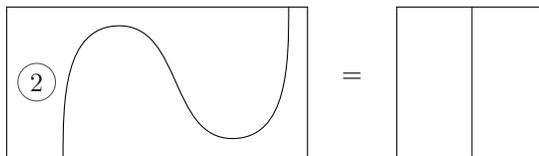
- d. Using either matrix arithmetic or algebra (your choice) verify that

$$[[Z_1^2 \cdot Z_0^1]] = [[Z_0^2]]$$

i.e.:



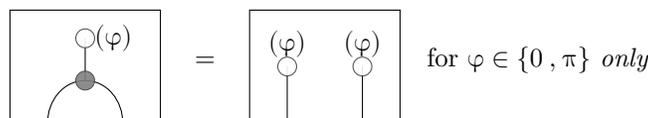
- e. Use the equations above, together with those of a *Frobenius algebra* to show by diagram rewriting that the following *zigzag law* holds in the ZX-calculus:



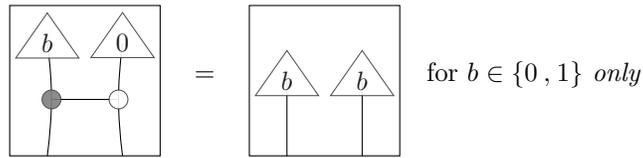
i.e. $2(\cap \otimes \text{Id}) \cdot (\text{Id} \otimes \cup) = \text{Id}$. *Note:* In the reading, the bent-wire state (\cap) and effect (\cup) are not normalized, so the scalars work out slightly differently there.

Problem 3 (Z-basis copying)

Recall that the spider Z_2^1 can copy Z-basis states:



While studying the *no-cloning theorem* we observed that the CNOT operator can also be used to copy Z-basis states:



- Either give a precise argument appealing to the properties of linear maps or use matrix arithmetic (your choice) to show that the interpretation of the operator $(\text{Id} \otimes |0\rangle) \cdot \text{CNOT}$ is the same as that of the spider Z_2^1 .
- Translate the circuit diagram for $(\text{Id} \otimes |0\rangle) \cdot \text{CNOT}$ into spider notation and use the rules of the ZX-calculus to rewrite it to the spider Z_2^1 .

Problem 4 (The GHZ state)

After the Bell states, one of the most important entangled states is the **GHZ state**:

$$|\text{GHZ}\rangle := \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)$$

- What is the spider notation for this state (you may ignore the scalar)?
- Use the rules of ZX-calculus to rewrite the GHZ state as a composition of the spiders Z_2^1 and Z_2^0 (you may again ignore the scalars).
- Using your result from problem 3 (b), translate your ZX-calculus diagram from part (b) into a circuit diagram. You may want to check your results either using matrix arithmetic or by running your circuit on the IBM quantum computer simulator at <https://quantumexperience.ng.bluemix.net/qstage/>.