Homework 5:
Duality and Super-Dense Coding

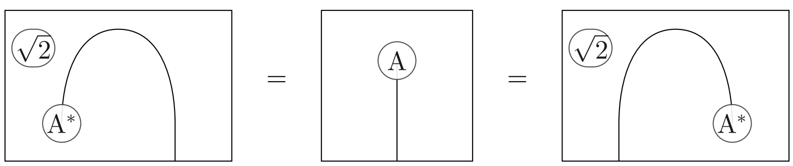
Quantum Information Systems Wesleyan University

due 2017.04.19

Problem 1 (duals of states)
In class we analyzed what would happen in the *super-dense coding protocol* if Bob were to observe his qubit prior to measuring it together with Alice's in the Bell basis. In that analysis, we used (one of) the following facts: for a qubit state $|\mathrm{A}\rangle$,

$$\sqrt{2}\,\left(|\cap\rangle \cdot (|\mathrm{A}\rangle^* \otimes \mathrm{Id})\right) \qquad = \qquad |\mathrm{A}\rangle \qquad = \qquad \sqrt{2}\,\left(|\cap\rangle \cdot (\mathrm{Id} \otimes |\mathrm{A}\rangle^*)\right)$$

or diagrammatically:



Note: in class I neglected the $\sqrt{2}$ scalar, my apologies. Also, in these diagrams I have drawn beads rather than triangles for the states in order to emphasize that the "upside down" ones are the duals, $|\mathrm{A}\rangle^*$, and not (necessarily) the adjoints, $\langle\mathrm{A}|$.

In this problem you will verify these two equations. Suppose the qubit state $|\mathrm{A}\rangle$ has matrix representation:

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

a. Use matrix arithmetic to verify that $\sqrt{2}\,\left(|\cap\rangle \cdot (|\mathrm{A}\rangle^* \otimes \mathrm{Id})\right) \;=\; |\mathrm{A}\rangle$

b. Use matrix arithmetic to verify that $\sqrt{2}\,\left(|\cap\rangle \cdot (\mathrm{Id} \otimes |\mathrm{A}\rangle^*)\right) \;=\; |\mathrm{A}\rangle$

Problem 2 (why Z- and X- negation have funny names—algebraically)
I've been promising you for a while now that we'd eventually discover why the Z-basis negation operator is called "X", while the X-basis negation operator is called "Z". Now's the time!

a. Use the facts that:

$$|+\rangle \;=\; \frac{1}{\sqrt{2}} \bullet (|0\rangle + |1\rangle) \qquad \text{and} \qquad |-\rangle \;=\; \frac{1}{\sqrt{2}} \bullet (|0\rangle - |1\rangle)$$

(here, I use "$- \bullet -$" for scalar multiplication for emphasis, although simple juxtaposition is also common notation) together with the (right) *distributive law* for composition over superposition:

$$(\mathrm{A} + \mathrm{B}) \cdot \mathrm{F} \quad = \quad (\mathrm{A} \cdot \mathrm{F}) + (\mathrm{B} \cdot \mathrm{F})$$

and the *interchange law* for scalar multiplication and composition:

$$(z_1 \bullet \mathrm{F}) \cdot (z_2 \bullet \mathrm{G}) \quad = \quad (z_1 z_2) \bullet (\mathrm{F} \cdot \mathrm{G})$$

to show *algebraically* (i.e. using 1-dimensional equational reasoning, *not* matrix arithmetic) that:

$$|+\rangle \cdot \mathrm{X} \;=\; |+\rangle \qquad \text{and} \qquad |-\rangle \cdot \mathrm{X} \;=\; -\,(|-\rangle)$$

b. Explain why this is sufficient to conclude that $\mathrm{X} = \mathrm{X}(\pi)$; i.e., that Z-basis negation is the same operation as X-basis phase rotation by π.

c. Use the distributivity of scalar multiplication over superposition:

$$z \bullet (\mathrm{A} + \mathrm{B}) \quad = \quad (z \bullet \mathrm{A}) + (z \bullet \mathrm{B})$$

together with superposition associativity and commutativity to verify *algebraically* that:

$$\frac{1}{\sqrt{2}} \bullet (|+\rangle + |-\rangle) \;=\; |0\rangle \qquad \text{and} \qquad \frac{1}{\sqrt{2}} \bullet (|+\rangle - |-\rangle) \;=\; |1\rangle$$

By a calculation nearly identical to the one that you did for part (a) (which I will not make you do here), you can show that:

$$|0\rangle \cdot \mathrm{Z} \quad = \quad |0\rangle \qquad \text{and} \qquad |1\rangle \cdot \mathrm{Z} \quad = \quad -(|1\rangle)$$

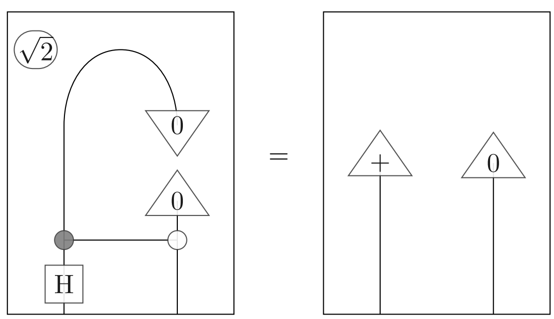
and thus that $\mathrm{Z} = \mathrm{Z}(\pi)$; i.e., that X-basis negation is the same operation as Z-basis phase rotation by π.

Problem 3 (Bob eats the marshmallow)
Recall that in the super-dense coding protocol, if Alice wants to send Bob a message consisting of the ordered pair of bits $(a\,,b)$ then she applies the following unitary operator to her qubit before sending it on to Bob:

$$\mathrm{U} \quad := \quad \mathrm{X}^b \cdot \mathrm{Z}^a$$

where we pun on exponentiating by a bit as a shorthand for $\mathrm{F}^0 := \mathrm{Id}$ and $\mathrm{F}^1 := \mathrm{F}$ for any operator F.

a. In class we used a simple diagram rewriting argument to show that if Alice tries to send the message $(0\,,0)$, but Bob peeks at his qubit and finds the state $|0\rangle$ before applying the $\mathrm{BELL}^\dagger$ operator, then he will receive the state $|+\,,0\rangle$ instead of the intended state $|0\,,0\rangle$:



Verify this result using matrix arithmetic:

$$\sqrt{2}\;(|\cap\rangle \cdot (\mathrm{Id} \otimes \langle 0|) \cdot (\mathrm{Id} \otimes |0\rangle) \cdot \mathrm{CNOT} \cdot (\mathrm{H} \otimes \mathrm{Id})) \quad = \quad |+\rangle \otimes |0\rangle$$

b. In class we worked out four of the eight possible cases for the state Bob would receive if he peeks at his qubit (in the Z-basis). Use diagram rewriting to work out the remaining four cases and complete the following table:

Alice sends	Bob peeks	Bob receives
$(0\,,0)$	$\|0\rangle$	$\|+\,,0\rangle$
$(0\,,0)$	$\|1\rangle$	$\|-\,,0\rangle$
$(0\,,1)$	$\|0\rangle$	?
$(0\,,1)$	$\|1\rangle$	?
$(1\,,0)$	$\|0\rangle$	$\|+\,,0\rangle$
$(1\,,0)$	$\|1\rangle$	$-(\|-\,,0\rangle)$
$(1\,,1)$	$\|0\rangle$	?
$(1\,,1)$	$\|1\rangle$	?

If you're feeling industrious, you may optionally check your results against the IBM quantum computer simulator at `https://quantumexperience.ng.bluemix.net/qstage/` (*hint*: to get a better look at the left qubit, measure it in the X-basis)

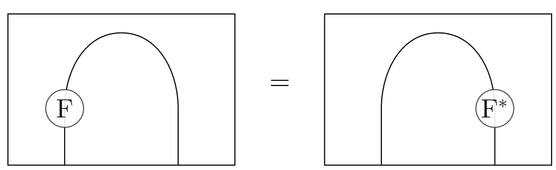
c. Bob notices that for the above table, in each case the left qubit he receives is (a phase multiple of) an X-basis state. He decides to be clever and measure the left qubit in the X-basis rather than in the Z-basis after applying the $\text{BELL}^\dagger$ operator. Does this allow him to distinguish any more of Alice's four possible messages than he could by measuring both qubits in the Z-basis? Why or why not?

d. Bob has another great idea: maybe he can peek at his qubit in the X-basis rather than in the Z-basis before applying the $\text{BELL}^\dagger$ operator without breaking the super-dense coding protocol. Fill out the following table to determine whether his plan will work (you need not submit your calculations):

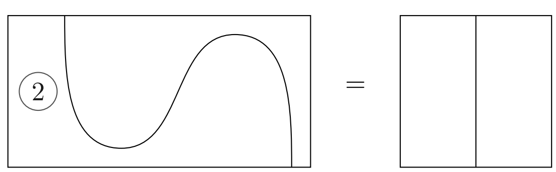
Alice sends	Bob peeks	Bob receives
$(0\,,0)$	$\|+\rangle$	?
$(0\,,0)$	$\|-\rangle$	?
$(0\,,1)$	$\|+\rangle$	?
$(0\,,1)$	$\|-\rangle$	?
$(1\,,0)$	$\|+\rangle$	?
$(1\,,0)$	$\|-\rangle$	?
$(1\,,1)$	$\|+\rangle$	?
$(1\,,1)$	$\|-\rangle$	?

How does the information that Bob is able to salvage in this case differ from the case where he peeks in the Z-basis?

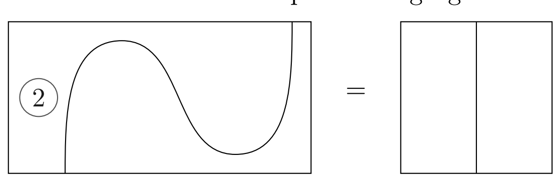
Problem 4 (zigzags and duals)
In class we used matrix arithmetic to show that "flipping a bead" over the bent wire state $|\cap\rangle$ gives us its dual:



We also showed that the following *zigzag law* holds:



a. Use matrix arithmetic to show that the other possible zigzag law also holds:



b. Use these equations, together with the fact that duality is an involution ($(\text{F}^*)^* = \text{F}$) to show the following by diagram rewriting: