Problem 1 (duals of states)

In class we analyzed what would happen in the super-dense coding protocol if Bob were to observe his qubit prior to measuring it together with Alice’s in the Bell basis. In that analysis, we used (one of) the following facts: for a qubit state $|A\rangle$,

$$\sqrt{2} \left( |\cap\rangle \cdot (|A\rangle^* \otimes \text{Id}) \right) = |A\rangle = \sqrt{2} \left( |\cap\rangle \cdot (\text{Id} \otimes |A\rangle^*) \right)$$

or diagrammatically:

Note: in class I neglected the $\sqrt{2}$ scalar, my apologies. Also, in these diagrams I have drawn beads rather than triangles for the states in order to emphasize that the “upside down” ones are the duals, $|A\rangle^*$, and not (necessarily) the adjoints, $\langle A|$.

In this problem you will verify these two equations. Suppose the qubit state $|A\rangle$ has matrix representation:

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

a. Use matrix arithmetic to verify that $\sqrt{2} \left( |\cap\rangle \cdot (|A\rangle^* \otimes \text{Id}) \right) = |A\rangle$

b. Use matrix arithmetic to verify that $\sqrt{2} \left( |\cap\rangle \cdot (\text{Id} \otimes |A\rangle^*) \right) = |A\rangle$

Problem 2 (why Z- and X- negation have funny names—algebraically)

I’ve been promising you for a while now that we’d eventually discover why the Z-basis negation operator is called “X”, while the X-basis negation operator is called “Z”. Now’s the time!

a. Use the facts that:

$$|+\rangle = \frac{1}{\sqrt{2}} \cdot (|0\rangle + |1\rangle) \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}} \cdot (|0\rangle - |1\rangle)$$

(here, I use “$\cdot$” for scalar multiplication for emphasis, although simple juxtaposition is also common notation) together with the (right) distributive law for composition over superposition:

$$(A + B) \cdot F = (A \cdot F) + (B \cdot F)$$

and the interchange law for scalar multiplication and composition:

$$(z_1 \bullet F) \cdot (z_2 \bullet G) = (z_1 z_2) \bullet (F \cdot G)$$

to show algebraically (i.e. using 1-dimensional equational reasoning, not matrix arithmetic) that:

$$|+\rangle \cdot X = |+\rangle \quad \text{and} \quad |-\rangle \cdot X = -(|-\rangle)$$
b. Explain why this is sufficient to conclude that $X = X(\pi)$; i.e., that $Z$-basis negation is the same operation as $X$-basis phase rotation by $\pi$.

c. Use the distributivity of scalar multiplication over superposition:

$$z \cdot (A + B) = (z \cdot A) + (z \cdot B)$$

together with superposition associativity and commutativity to verify algebraically that:

$$\frac{1}{\sqrt{2}} \cdot (|+\rangle + |-\rangle) = |0\rangle \quad \text{and} \quad \frac{1}{\sqrt{2}} \cdot (|+\rangle - |-\rangle) = |1\rangle$$

By a calculation nearly identical to the one that you did for part (a) (which I will not make you do here), you can show that:

$$|0\rangle \cdot Z = |0\rangle \quad \text{and} \quad |1\rangle \cdot Z = -|1\rangle$$

and thus that $Z = Z(\pi)$; i.e., that $X$-basis negation is the same operation as $Z$-basis phase rotation by $\pi$.

**Problem 3 (Bob eats the marshmallow)**

Recall that in the superdense coding protocol, if Alice wants to send Bob a message consisting of the ordered pair of bits $(a, b)$ then she applies the following unitary operator to her qubit before sending it on to Bob:

$$U := X^b \cdot Z^a$$

where we pun on exponentiating by a bit as a shorthand for $F^0 := \text{Id}$ and $F^1 := F$ for any operator $F$.

a. In class we used a simple diagram rewriting argument to show that if Alice tries to send the message $(0, 0)$, but Bob peeks at his qubit and finds the state $|0\rangle$ before applying the $\text{BELL}^+$ operator, then he will receive the state $|+, 0\rangle$ instead of the intended state $|0, 0\rangle$:

Verify this result using matrix arithmetic:

$$\sqrt{2} (|\cap\rangle \cdot (\text{Id} \otimes |0\rangle) \cdot (\text{Id} \otimes |0\rangle) \cdot \text{CNOT} \cdot (H \otimes \text{Id})) = |+\rangle \otimes |0\rangle$$

b. In class we worked out four of the eight possible cases for the state Bob would receive if he peeks at his qubit (in the $Z$-basis). Use diagram rewriting to work out the remaining four cases and complete the following table:

<table>
<thead>
<tr>
<th>Alice sends</th>
<th>Bob peeks</th>
<th>Bob receives</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 0)$</td>
<td>$</td>
<td>0\rangle$</td>
</tr>
<tr>
<td>$(0, 0)$</td>
<td>$</td>
<td>1\rangle$</td>
</tr>
<tr>
<td>$(0, 1)$</td>
<td>$</td>
<td>0\rangle$</td>
</tr>
<tr>
<td>$(0, 1)$</td>
<td>$</td>
<td>1\rangle$</td>
</tr>
<tr>
<td>$(1, 0)$</td>
<td>$</td>
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</tr>
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<td>$(1, 0)$</td>
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</tr>
<tr>
<td>$(1, 1)$</td>
<td>$</td>
<td>1\rangle$</td>
</tr>
</tbody>
</table>
If you’re feeling industrious, you may optionally check your results against the IBM quantum computer simulator at https://quantumexperience.ng.bluemix.net/qstage/ (hint: to get a better look at the left qubit, measure it in the X-basis)

c. Bob notices that for the above table, in each case the left qubit he receives is (a phase multiple of) an X-basis state. He decides to be clever and measure the left qubit in the X-basis rather than in the Z-basis after applying the BELL† operator. Does this allow him to distinguish any more of Alice’s four possible messages than he could by measuring both qubits in the Z-basis? Why or why not?

d. Bob has another great idea: maybe he can peek at his qubit in the X-basis rather than in the Z-basis before applying the BELL† operator without breaking the super-dense coding protocol. Fill out the following table to determine whether his plan will work (you need not submit your calculations):

<table>
<thead>
<tr>
<th>Alice sends</th>
<th>Bob peeks</th>
<th>Bob receives</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>+</td>
<td>?</td>
</tr>
<tr>
<td>(0, 0)</td>
<td>−</td>
<td>?</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>+</td>
<td>?</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>−</td>
<td>?</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>+</td>
<td>?</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>−</td>
<td>?</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>+</td>
<td>?</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>−</td>
<td>?</td>
</tr>
</tbody>
</table>

How does the information that Bob is able to salvage in this case differ from the case where he peeks in the Z-basis?

Problem 4 (zigzags and duals)
In class we used matrix arithmetic to show that “flipping a bead” over the bent wire state $|\cap\rangle$ gives us its dual:

$$ F = F^\ast $$

We also showed that the following zigzag law holds:

$$ 2 = 2 $$

a. Use matrix arithmetic to show that the other possible zigzag law also holds:

$$ 2 = 2 $$

b. Use these equations, together with the fact that duality is an involution $(F^\ast)^\ast = F$ to show the following by diagram rewriting:

$$ F^\ast = 2 $$