

Homework 4: The Z-, X- and Bell Bases

Quantum Information Systems Wesleyan University

due 2017.04.12

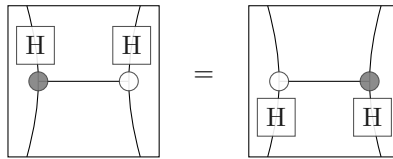
Problem 1 (Hadamard meets CNOT)

In a sense that will become clear later, the Hadamard and controlled negation operators mediate between the Z- and X- bases. In this problem, we explore some interactions between them.

a. Use matrix arithmetic to verify the following interaction law:

$$(H \otimes H) \cdot \text{CNOT} = \text{TONC} \cdot (H \otimes H)$$

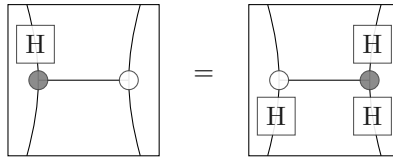
or diagrammatically:



b. Use the equation from part (a) and diagram rewriting to show that:

$$(H \otimes \text{Id}) \cdot \text{CNOT} = (\text{Id} \otimes H) \cdot \text{TONC} \cdot (H \otimes H)$$

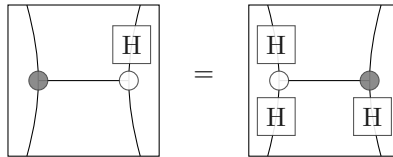
i.e.



c. Likewise, show that:

$$(\text{Id} \otimes H) \cdot \text{CNOT} = (H \otimes \text{Id}) \cdot \text{TONC} \cdot (H \otimes H)$$

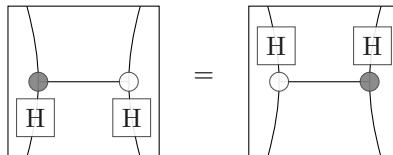
i.e.



d. And that:

$$\text{CNOT} \cdot (H \otimes H) = (H \otimes H) \cdot \text{TONC}$$

i.e.



Problem 2 (CNOT in the X-basis)

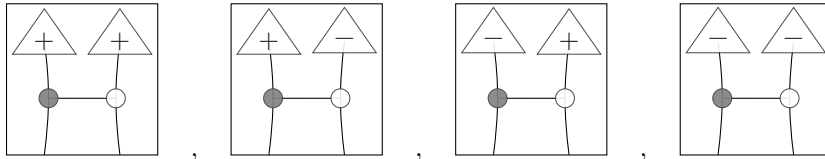
From the perspective of the Z-basis states, it looks as though the CNOT operator acts like the identity function when restricted to the left qubit and that the value of the left qubit is “controlling” the negation of the right qubit. However, these intuitions, which are true of boolean *bits*, are not actually correct when it comes to *qubits*.

Recall that when we introduced CNOT and its symmetric version TONC, we presented them by the following “truth tables” in the Z-basis:

A	B	A•→B	
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

A	B	A↔•B	
0	0	0	0
0	1	1	1
1	0	1	0
1	1	0	1

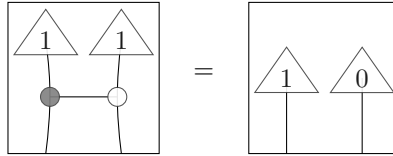
- a. Use rewriting to simplify each of the following diagrams to a tensor product of states (i.e. to diagrams involving no single- or multi-qubit operators):



Hint: each row in the above “truth tables” encodes a rewrite rule. For example, the bottom row of the table for CNOT encodes:

$$(|1\rangle \otimes |1\rangle) \cdot \text{CNOT} = |1\rangle \otimes |0\rangle$$

or diagrammatically:



- b. Use your results from part (a) to complete the following “truth table” for CNOT in the X-basis:

A	B	A•→B	
+	+	?	?
+	-	?	?
-	+	?	?
-	-	?	?

- c. From the perspective of the X-basis states, on which qubit does the CNOT operator appear to act as the identity function? Which qubit appears to be “controlling” the other?

Problem 3 (CNOT and X-basis negation)

In class we met the *X-basis negation operator*:

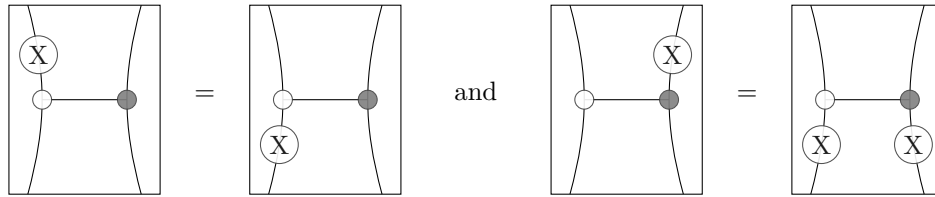
$$Z = |-\rangle\langle+| + |+\rangle\langle-|$$

and showed that $Z = H \cdot X \cdot H$, where X is the familiar *Z-basis negation operator*:

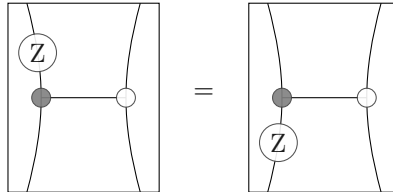
$$X = |1\rangle\langle 0| + |0\rangle\langle 1|$$

(I promise that there is a good reason for this backward-seeming naming scheme.)

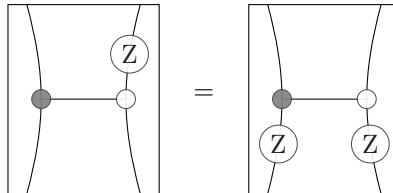
We also saw how Z-basis negation interacts with CNOT, or equivalently, with TONC:



- a. Use these facts together with those that you proved in problem 1 to show the following by rewriting:



- b. Similarly, show:



Problem 4 (the Bell basis)

We also met the **Bell basis** for two qubits, $(|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle)$, where:

$$\begin{aligned} |\beta_{00}\rangle &= \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \\ |\beta_{01}\rangle &= \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \\ |\beta_{10}\rangle &= \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) \\ |\beta_{11}\rangle &= \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \end{aligned}$$

We asserted that the Bell basis is orthonormal. In this problem you will check this assertion.

- You should check that each Bell basis vector has *unit norm*, but since all the cases are quite similar, for the purposes of this problem you need only check that $\|\beta_{00}\rangle\| = 1$.
- You should check that the Bell basis vectors are *pairwise orthogonal*, but since all the cases are quite similar, for the purposes of this problem you need only check that $|\beta_{00}\rangle \perp |\beta_{11}\rangle$.
- Check that the Bell basis vectors span the vector space $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Hint: it suffices to check that each of the standard basis vectors $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$ can be expressed as a linear combination of the Bell basis vectors. This is because we already know that those vectors span $\mathbb{C}^2 \otimes \mathbb{C}^2$.