Homework 1:  
States and Measurements

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**Problem 1** Recall that any orthonormal basis \((b_0, \ldots, b_{n-1})\) has the *identity decomposition property*:

\[
\sum_i |b_i\rangle \langle b_i| = \text{Id}_n
\]

Confirm via explicit calculation that the X-basis \((|+\rangle, |--\rangle)\) indeed has this property.

**Problem 2** Compute the probability of a measurement on the state \(|--\rangle\) resulting in the outcome corresponding to each of the vectors in the standard basis \((|0\rangle, |1\rangle)\).

**Problem 3** Recall that orthonormal bases \((D, E)\) for a vector space are complementary (or mutually unbiased) if the elements of each are indistinguishable by a measurement made in the other; i.e. if:

\[
\langle d_i | e_j \rangle \langle e_j | d_i \rangle
\]

is the same (real number \(0 \leq \rho \leq 1\)) for all \(d_i \in D\) and \(e_j \in E\).

Suppose \(\mathbb{V}\) is an \(n\)-dimensional vector space with complementary orthonormal bases \(D\) and \(E\). For \(0 \leq i < n\) and \(0 \leq j < n\), what is the probability of basis state \(|d_i\rangle \in D\) resulting in the measurement outcome corresponding to basis state \(|e_j\rangle \in E\)? Please explain your answer.

**Problem 4** We have met the Z-basis \((|0\rangle, |1\rangle)\) and the X-basis \((|+\rangle, |--\rangle)\), which are complementary. Another important basis for \(\mathbb{C}^2\) is the Y-basis:

\[
|\uparrow\rangle := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad |\downarrow\rangle := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}
\]

a. Compute the norm of each Y-basis vector.

b. Show that the Y-basis vectors are orthogonal \((|\uparrow\rangle \perp |\downarrow\rangle)\).

c. Show that the X-basis and Y-basis are complementary.

d. Show that the Y-basis and Z-basis are complementary.

Conclude that the X-, Y-, and Z-bases are three (pairwise) complementary orthonormal bases for \(\mathbb{C}^2\).