Homework 1:
States and Measurements

Quantum Information Systems   Wesleyan University

due 2017.03.06

Problem 1 Recall that any orthonormal basis $(b_0, \cdots, b_{n-1})$ has the *identity decomposition property*:

$$\sum_i |b_i\rangle\langle b_i| \quad = \quad \mathrm{Id}_n$$

Confirm via explicit calculation that the X-basis $(|+\rangle, |-\rangle)$ indeed has this property.

Problem 2 Compute the probability of a measurement on the state $|-\rangle$ resulting in the outcome corresponding to each of the vectors in the standard basis $(|0\rangle, |1\rangle)$.

Problem 3 Recall that orthonormal bases (D and E) for a vector space are *complementary* (or *mutually unbiased*) if the elements of each are indistinguishable by a measurement made in the other; i.e. if:

$$\langle d_i \mid e_j\rangle\langle e_j \mid d_i\rangle$$

is the same (real number $0 \leq p \leq 1$) for all $d_i \in \mathrm{D}$ and $e_j \in \mathrm{E}$.

Suppose $\mathbb{V}$ is an n-dimensional vector space with complementary orthonormal bases D and E. For $0 \leq i < n$ and $0 \leq j < n$, what is the probability of basis state $|d_i\rangle \in \mathrm{D}$ resulting in the measurement outcome corresponding to basis state $|e_j\rangle \in \mathrm{E}$? Please explain your answer.

Problem 4 We have met the Z-basis $(|0\rangle, |1\rangle)$ and the X-basis $(|+\rangle, |-\rangle)$, which are complementary. Another important basis for $\mathbb{C}^2$ is the Y-basis:

$$|\uparrow\rangle \quad := \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$|\downarrow\rangle \quad := \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

a. Compute the norm of each Y-basis vector.

b. Show that the Y-basis vectors are orthogonal $(|\uparrow\rangle \perp |\downarrow\rangle)$.

c. Show that the X-basis and Y-basis are complementary.

d. Show that the Y-basis and Z-basis are complementary.

Conclude that the X-, Y-, and Z-bases are three (pairwise) complementary orthonormal bases for $\mathbb{C}^2$.