Like simplicial sets, cubical sets provide a combinatorial model of homotopy theory.

However, there are several varieties of cubical sets to choose from.

Maps include faces, degeneracies, diagonals, connections, etc..

Relations witness properties of geometric cubes.
Like *simplicial sets*, *cubical sets* provide a combinatorial model of homotopy theory.

However, there are several varieties of cubical sets to choose from.

Maps include *faces*, *degeneracies*, *diagonals*, *connections*, etc..

Relations witness properties of *geometric cubes*.

Various criteria for choosing a cubical theory, including:

▶ homotopy theory (strict test categories),
▶ computational behavior (canonical forms, $x$-Reedy structure, distributive laws),
▶ model structure (judgemental vs typal equalities),
▶ etc.
Motivated by order-theoretic and monoidal structure, we present a simple cube category that:

- contains all the familiar maps,
Overview

Motivated by order-theoretic and monoidal structure, we present a simple cube category that:

- contains all the familiar maps,
- has a strong equational theory,
Motivated by order-theoretic and monoidal structure, we present a simple cube category that:

- contains all the familiar maps,
- has a strong equational theory,
- is a strict test category,
Motivated by order-theoretic and monoidal structure, we present a simple cube category that:

- contains all the familiar maps,
- has a strong equational theory,
- is a strict test category,
- is closely related to simplices.
Combinatorial Aspects
Simplicies, Order-Theoretically

An \( n \text{-simplex} \), \( \langle n \rangle \), is the walking path of \( n \) serially composable arrows.

An \( n \text{-simplex} \), \( \langle n \rangle \), is the walking path of \( n \) serially composable arrows.
Simplicies, Order-Theoretically

An $n$-simplex, “$\langle n \rangle$”, is the walking path of $n$ serially composable arrows.

The simplex category, “$\Delta$”, can be presented as the (skeleton of the) full subcategory of $\text{ORD}$ containing inhabited, finite, totally ordered sets:

\[
\langle n \rangle := \text{fin}(n + 1) \quad \text{e.g.} \quad \langle 2 \rangle := \{0, 1, 2\}
\]
Simplicies, Order-Theoretically

An \( n \)-simplex, “\( \langle n \rangle \)”, is the walking path of \( n \) serially composable arrows.

The simplex category, “\( \Delta \)”, can be presented as the (skeleton of the) full subcategory of \( \text{Ord} \) containing inhabited, finite, totally ordered sets:

\[
\langle n \rangle := \text{fin}(n + 1) \quad \text{e.g.} \quad \langle 2 \rangle := \{0, 1, 2\}
\]

Its maps are generated by:

- **faces** (dimension-raising maps) injective monotone functions
  
  e.g. \( d^1 = [0, 2] = \{0, 1\} \hookrightarrow \{0, 2\} : \Delta (\langle 1 \rangle \rightarrow \langle 2 \rangle) \)

- **degeneracies** (dimension-lowering maps) surjective monotone functions
  
  e.g. \( s^1 = [0, 1, 1] = \{0, 1, 2\} \twoheadrightarrow \{0, 1, 1\} : \Delta (\langle 2 \rangle \rightarrow \langle 1 \rangle) \)
Simplicies, Monoidally

The simplex category can also be presented via the *walking monoid*, which is the category $\mathbb{M}$ with:

- one generating object, $V : \mathbb{M}$
- two generating morphisms, $s : \mathbb{M} (V \otimes V \to V)$ and $d : \mathbb{M} (I \to V)$
- relations that make $(V, d, s)$ a monoid in $(\mathbb{M}, \otimes, I)$.

Then $\Delta$ is the full subcategory of $\mathbb{M}$ excluding the object $V^{\otimes 0}$ with $\langle n \rangle := V^{\otimes (n+1)}$. 
Simplicies, Monoidally

The simplex category can also be presented via the *walking monoid*, which is the category $\mathbb{M}$ with:

- one generating object, $V : \mathbb{M}$
- two generating morphisms, $s : \mathbb{M} (V \otimes V \rightarrow V)$ and $d : \mathbb{M} (I \rightarrow V)$
- relations that make $(V, d, s)$ a monoid in $(\mathbb{M}, \otimes, I)$.

Then $\Delta$ is the full subcategory of $\mathbb{M}$ excluding the object $V^{\otimes 0}$ with $\langle n \rangle := V^{\otimes (n+1)}$.

Example: composing $d^1 : \Delta (\langle 1 \rangle \rightarrow \langle 2 \rangle)$ with $s^1 : \Delta (\langle 2 \rangle \rightarrow \langle 1 \rangle)$:
Ordered (Monoidal) Cubes?

The well-studied cube categories also have order-theoretic [Jar06] and monoidal [GM03] presentations.

But in the monoidal presentation there is a “dimension mismatch”: the generating object is an *interval* rather than a *point*.
Ordered (Monoidal) Cubes?

The well-studied cube categories also have order-theoretic [Jar06] and monoidal [GM03] presentations.

But in the monoidal presentation there is a “dimension mismatch”: the generating object is an *interval* rather than a *point*.

Goal: a *vertex-based* cube category with all familiar maps and relations that is related to the simplex category by their order-theoretic presentations.
Ordered Cubes

The **standard geometric** $n$-cube is the convex subspace of $\mathbb{R}^n$ bounded by the $2^n$ vertex points $v = (v_0, \ldots, v_{n-1})$ where $v_i \in \{0, 1\}$.
Ordered Cubes

The standard geometric $n$-cube is the convex subspace of $\mathbb{R}^n$ bounded by the $2^n$ vertex points $v = (v_0, \ldots, v_{n-1})$ where $v_i \in \{0, 1\}$.

Therefore we define:

Definition

An ordered $n$-cube, “[n]”, is the preorderd set $\{0 \leq 1\}^n$.
Ordered Cubes

The **standard geometric** $n$-cube is the convex subspace of $\mathbb{R}^n$ bounded by the $2^n$ vertex points $v = (v_0, \ldots, v_{n-1})$ where $v_i \in \{0, 1\}$.

Therefore we define:

**Definition**

An **ordered** $n$-cube, “$[n]$”, is the preorder set $\{0 \leq 1\}^\times n$

- $[n]$ is the walking product of $n$ arrows.
Ordered Cubes

The standard geometric $n$-cube is the convex subspace of $\mathbb{R}^n$ bounded by the $2^n$ vertex points $v = (v_0, \ldots, v_{n-1})$ where $v_i \in \{0, 1\}$.

Therefore we define:

**Definition**

An ordered $n$-cube, “$[n]$”, is the preorderd set $\{0 \leq 1\}^n$

- $[n]$ is the walking product of $n$ arrows.
- Each $[n]$ is a complete and distributive lattice.
Ordered Cubes

The **standard geometric** \( n \)-**cube** is the convex subspace of \( \mathbb{R}^n \) bounded by the \( 2^n \) vertex points \( v = (v_0, \ldots, v_{n-1}) \) where \( v_i \in \{0, 1\} \).

Therefore we define:

**Definition**

An **ordered** \( n \)-**cube**, “\([n]\)”, is the preorder set \( \{0 \leq 1\}^n \)

- \([n]\) is the walking product of \( n \) arrows.
- Each \([n]\) is a complete and distributive lattice.
- \([n]\) is isomorphic to the subset lattice of \( \text{fin}(n) \) where \( v_i = 1 \iff i \in v \):

\[
\begin{align*}
000 &\rightarrow 010 &\rightarrow 001 &\rightarrow 011 &\rightarrow 111 \\
100 &\rightarrow 110 &\rightarrow 101 &\rightarrow 111
\end{align*}
\]

\[
\begin{align*}
\emptyset &\rightarrow \{0\} &\rightarrow \{0, 1\} &\rightarrow \{0, 1, 2\} \\
\{0\} &\rightarrow \{0, 1\} &\rightarrow \{0, 1, 2\}
\end{align*}
\]
**Definition**

The **ordered cube category**, “□”, is the full subcategory of $\text{ORD}$ (thus of $\text{CAT}$) containing the ordered cubes.
Ordered Cube Category

Definition
The ordered cube category, “□”, is the full subcategory of $\text{ORD}$ (thus of $\text{CAT}$) containing the ordered cubes.

Among its maps are the:

- **aspects** (dimension-raising maps) injective monotone functions
  $$\square ([n - 1] \rightarrow [n])$$

- **derivatives** (dimension-lowering maps) surjective monotone functions
  $$\square ([n + 1] \rightarrow [n])$$
Familiar Aspects

Aspects include:

Inserting coordinate $b \in \{0, 1\}$ at index $i$ of every vertex gives a map $[i \mapsto b] : \square ([n - 1] \to [n])$ determining a face.

Although drawn as polytopes, these are just order-preserving maps of vertices.
Familiar Aspects

Aspects include:

Inserting coordinate $b \in \{0, 1\}$ at index $i$ of every vertex gives a map $[i \mapsto b] : \square ([n - 1] \rightarrow [n])$ determining a face.

Although drawn as polytopes, these are just order-preserving maps of vertices.
Familiar Aspects

Aspects include:

Inserting coordinate $b \in \{0, 1\}$ at index $i$ of every vertex gives a map $[i \mapsto b] : \square ([n - 1] \rightarrow [n])$ determining a face.

Although drawn as polytopes, these are just order-preserving maps of vertices.
Familiar Aspects

Aspects include:

Inserting coordinate $b \in \{0, 1\}$ at index $i$ of every vertex gives a map $[i \mapsto b] : \square ([n - 1] \rightarrow [n])$ determining a face.

Inserting a copy of the coordinate in index $i$ at index $j$ of every vertex (where $i < j$) gives a map $\delta(i, j) : \square ([n - 1] \rightarrow [n])$, determining a diagonal.
Familiar Aspects

Aspects include:

Inserting coordinate $b \in \{0, 1\}$ at index $i$ of every vertex gives a map $[i \mapsto b] : \Box ([n - 1] \to [n])$ determining a face.

Inserting a copy of the coordinate in index $i$ at index $j$ of every vertex (where $i < j$) gives a map $\delta(i, j) : \Box ([n - 1] \to [n])$, determining a diagonal.
Familiar Aspects

Aspects include:

Inserting coordinate $b \in \{0, 1\}$ at index $i$ of every vertex gives a map $[i \mapsto b] : \square ([n - 1] \to [n])$ determining a face.

Inserting a copy of the coordinate in index $i$ at index $j$ of every vertex (where $i < j$) gives a map $\delta(i, j) : \square ([n - 1] \to [n])$, determining a diagonal.
Familiar Aspects

Aspects include:

Inserting coordinate \( b \in \{0, 1\} \) at index \( i \) of every vertex gives a map \([i \mapsto b] : \square (\{n - 1\} \to \{n\})\) determining a face.

Inserting a copy of the coordinate in index \( i \) at index \( j \) of every vertex (where \( i < j \)) gives a map \( \delta(i, j) : \square (\{n - 1\} \to \{n\})\), determining a diagonal.

Although drawn as polytopes, these are just order-preserving maps of vertices.
Familiar Derivatives

Derivatives include:

Deleting the coordinate at index $i$ of every vertex gives a map

$$\hat{i} : \square ([n + 1] \rightarrow [n])$$

determining a degeneracy.
Familiar Derivatives

Derivatives include:

Deleting the coordinate at index $i$ of every vertex gives a map $\hat{i} : \square([n + 1] \to [n])$ determining a **degeneracy**.
Familiar Derivatives

Derivatives include:

Deleting the coordinate at index $i$ of every vertex gives a map

$$\hat{i} : \square ([n + 1] \to [n])$$

determining a \textbf{degeneracy}.
Familiar Derivatives

Derivatives include:

Deleting the coordinate at index $i$ of every vertex gives a map $\hat{i} : □ ([n + 1] \rightarrow [n])$ determining a **degeneracy**.

For each vertex $v$ and $* \in \{\vee, \wedge\}$, computing the coordinate $b := v_i * v_j$, then deleting the coordinates at indices $i$ and $j$, then inserting $b$ at index $k$ gives a map $[k\mapsto i * j] : □ ([n + 1] \rightarrow [n])$ determining a **connection**.
Derivatives include:

Deleting the coordinate at index $i$ of every vertex gives a map $\hat{i} : \square ([n + 1] \to [n])$ determining a degeneracy.

For each vertex $v$ and $* \in \{\lor, \land\}$, computing the coordinate $b := v_i \ast v_j$, then deleting the coordinates at indices $i$ and $j$, then inserting $b$ at index $k$ gives a map $[k \mapsto i \ast j] : \square ([n + 1] \to [n])$ determining a connection.
Familiar Derivatives

Derivatives include:

Deleting the coordinate at index $i$ of every vertex gives a map
$\hat{i} : \Box ([n + 1] \to [n])$ determining a **degeneracy**.

For each vertex $v$ and $\ast \in \{\lor, \land\}$, computing the coordinate $b := v_i \ast v_j$, then deleting the coordinates at indices $i$ and $j$, then inserting $b$ at index $k$ gives a map $[k \mapsto i \ast j] : \Box ([n + 1] \to [n])$ determining a **connection**.

Thus $\Box$ has the usual cubical maps.
But there are additional maps as well, for example, the “bent square” aspect of the cube:

\[
\begin{array}{c}
\begin{array}{c|c}
\beta & [3] \\
[2] & 000 \\
00 & 011 \\
01 & 101 \\
10 & 111 \\
11 & 110 \\
\end{array}
\end{array}
\]

Note: several workshop participants observed that this map is not, in fact, novel, and I am grateful to Ulrik Buchholtz for pointing out to me that the ordered cubes are equivalent to the distributive lattice cubes.
Since $\Delta \subseteq \text{ORD}$ and $\Box \subseteq \text{ORD}$, we can consider maps in the hom $\text{ORD}(\langle m \rangle \to [n])$.

It suffices to consider the nondegenerate (i.e. injective) maps in the hom $\text{ORD}(\langle n \rangle \to [n])$. 

This determines a triangulation profunctor $t : \Box \leftrightarrow \Delta$ (i.e. $\Delta^\circ \times \Box \to \text{Set}$).
Triangulation

Since $\Delta \subseteq \text{ORD}$ and $\Box \subseteq \text{ORD}$, we can consider maps in the hom $\text{ORD}(\langle m \rangle \rightarrow [n])$.

It suffices to consider the nondegenerate (i.e. injective) maps in the hom $\text{ORD}(\langle n \rangle \rightarrow [n])$.

Each permutation of $\text{fin}(n)$ corresponds to an ordered embedding $\langle n \rangle \hookrightarrow [n]$ by choosing an index (i.e. dimension) for each arrow in the path:
Triangulation

Since $\Delta \subseteq \text{ORD}$ and $\square \subseteq \text{ORD}$, we can consider maps in the hom $\text{ORD} (\langle m \rangle \to \{ n \})$.

It suffices to consider the nondegenerate (i.e. injective) maps in the hom $\text{ORD} (\langle n \rangle \to \{ n \})$.

Each permutation of $\text{fin} (n)$ corresponds to an ordered embedding $\langle n \rangle \hookrightarrow \{ n \}$ by choosing an index (i.e. dimension) for each arrow in the path:

$[0, 1, 2]$
Triangulation

Since $\triangle \subseteq \text{ORD}$ and $\square \subseteq \text{ORD}$, we can consider maps in the hom $\text{ORD}(\langle m \rangle \to \lfloor n \rfloor)$.

It suffices to consider the nondegenerate (i.e. injective) maps in the hom $\text{ORD}(\langle n \rangle \to \lfloor n \rfloor)$.

Each permutation of $\text{fin}(n)$ corresponds to an ordered embedding $\langle n \rangle \hookrightarrow \lfloor n \rfloor$ by choosing an index (i.e. dimension) for each arrow in the path:

$[0, 2, 1]$
Triangulation

Since $\Delta \subseteq \text{ORD}$ and $\Box \subseteq \text{ORD}$, we can consider maps in the hom $\text{ORD}(\langle m \rangle \to [n])$.

It suffices to consider the nondegenerate (i.e. injective) maps in the hom $\text{ORD}(\langle n \rangle \to [n])$.

Each permutation of $\text{fin}(n)$ corresponds to an ordered embedding $\langle n \rangle \hookrightarrow [n]$ by choosing an index (i.e. dimension) for each arrow in the path:

$[2, 0, 1]$

![Diagram showing a triangulation profunctor $t : \Box \to \Delta$]
Triangulation

Since $\Delta \subseteq \text{ORD}$ and $\Box \subseteq \text{ORD}$, we can consider maps in the hom $\text{ORD}(\langle m \rangle \rightarrow [n])$.

It suffices to consider the nondegenerate (i.e. injective) maps in the hom $\text{ORD}(\langle n \rangle \rightarrow [n])$.

Each permutation of $\text{fin}(n)$ corresponds to an ordered embedding $\langle n \rangle \hookrightarrow [n]$ by choosing an index (i.e. dimension) for each arrow in the path:

$[1, 0, 2]$
Since $\Delta \subseteq \text{ORD}$ and $\square \subseteq \text{ORD}$, we can consider maps in the hom $\text{ORD}(\langle m \rangle \to [n])$.

It suffices to consider the nondegenerate (i.e. injective) maps in the hom $\text{ORD}(\langle n \rangle \to [n])$.

Each permutation of $\text{fin}(n)$ corresponds to an ordered embedding $\langle n \rangle \hookrightarrow [n]$ by choosing an index (i.e. dimension) for each arrow in the path:

$[1, 2, 0]$
Triangulation

Since $\Delta \subseteq \text{ORD}$ and $\Box \subseteq \text{ORD}$, we can consider maps in the hom $\text{ORD}(\langle m \rangle \to [n])$.

It suffices to consider the nondegenerate (i.e. injective) maps in the hom $\text{ORD}(\langle n \rangle \to [n])$.

Each permutation of $\text{fin}(n)$ corresponds to an ordered embedding $\langle n \rangle \hookrightarrow [n]$ by choosing an index (i.e. dimension) for each arrow in the path:

$[2, 1, 0]$
Triangulation

Since $\Delta \subseteq \text{ORD}$ and $\square \subseteq \text{ORD}$, we can consider maps in the hom $\text{ORD} (\langle m \rangle \to [n]).$

It suffices to consider the nondegenerate (i.e. injective) maps in the hom $\text{ORD} (\langle n \rangle \to [n]).$

Each permutation of $\text{fin}(n)$ corresponds to an ordered embedding $\langle n \rangle \hookrightarrow [n]$ by choosing an index (i.e. dimension) for each arrow in the path:

$[2, 1, 0]$

This determines a triangulation profunctor $t : \square \to \Delta$ (i.e. $\Delta^\circ \times \square \to \text{SET}$).
Homotopical Aspects
Localization

For a category with weak equivalences \((\mathcal{C}, \mathcal{W})\) and a category \(\mathcal{D}\), any functor sending weak equivalences in \(\mathcal{C}\) to isos in \(\mathcal{D}\) factors through a localization functor sending weak equivalences to isos in the homotopy category of \(\mathcal{C}\).

\[(\mathcal{C}, \mathcal{W}) \xrightarrow{F} (\mathcal{D}, \mathcal{I})\]
Localization

For a category with weak equivalences \((\mathbb{C}, \mathcal{W})\) and a category \(\mathbb{D}\), any functor sending weak equivalences in \(\mathbb{C}\) to isos in \(\mathbb{D}\) factors through a localization functor sending weak equivalences to isos in the homotopy category of \(\mathbb{C}\).

\[
\begin{array}{ccc}
(\text{Ho } \mathbb{C}, \mathcal{J}) & \xrightarrow{\gamma \mathbb{C}} & (\text{Ho } F, \mathcal{J}) \\
\downarrow & & \downarrow \\
(\mathbb{C}, \mathcal{W}) & \xrightarrow{F} & (\mathbb{D}, \mathcal{J})
\end{array}
\]
Localization

For a category with weak equivalences \((\mathcal{C}, \mathcal{W})\) and a category \(\mathcal{D}\), any functor sending weak equivalences in \(\mathcal{C}\) to isos in \(\mathcal{D}\) factors through a localization functor sending weak equivalences to isos in the homotopy category of \(\mathcal{C}\).

\[
\begin{array}{ccc}
(\text{Ho } \mathcal{C}, \mathcal{I}) & \xrightarrow{\gamma \mathcal{C}} & (\text{Ho } F, \mathcal{I}) \\
\downarrow & & \downarrow \\
(\mathcal{C}, \mathcal{W}) & \xrightarrow{F} & (\mathcal{D}, \mathcal{I})
\end{array}
\]

The homotopy category can be constructed by freely adding inverses to the weak equivalences.
Test Categories

For small $\mathbb{S}$ and cocomplete $\mathbb{C}$, a functor $F : \mathbb{S} \to \mathbb{C}$ determines an adjunction where $\text{Lan}_y F(X) = \int^s : \mathbb{S} (X_s \otimes F_s)$

If this adjunction is an equivalence then $\mathbb{S}$ is a weak test category.

If this also holds true for all slices then $\mathbb{S}$ is a test category.

And if $\int_\mathbb{S} \cdot \gamma : \mathbb{Cat}$ preserves products then $\mathbb{S}$ is a strict test category.

We can do synthetic homotopy theory in the category of presheaves for any (strict) test category $\left[\text{Gro83}\right]$. 
Test Categories

The standard topological simplex functor determines geometric realization and singular complex.

\[
\begin{array}{ccc}
\Delta & \xrightarrow{y} & \Delta_{\text{TOP}} \\
\downarrow & & \downarrow \\
\hat{\Delta} & \xleftarrow{\text{sing}} & \text{TOP}
\end{array}
\]
Test Categories

The slice functor determines the category of elements and nerve (where $\int_S X = y(-)/X$).

\[
\begin{align*}
\hat{\mathcal{S}} & \xrightarrow{\int_S} \mathcal{S} & \mathcal{S}/- \\
\hat{\mathcal{S}} & \xleftarrow{\mathcal{N}_S} - \\
\mathcal{S} & \xrightarrow{\gamma} \mathcal{CAT}
\end{align*}
\]
Test Categories

Localization induces an adjunction on the homotopy categories.

\[ \hat{S} \xrightarrow{y} S \quad \xleftarrow{\gamma} \quad S/\neg \quad \xrightarrow{\int_S} \quad \perp \quad \xleftarrow{\Lan} \quad \text{CAT} \]

\[ \gamma \hat{S} \quad \xrightarrow{\int_S} \quad \perp \quad \xleftarrow{\Ran} \quad \text{Ho CAT} \]

\[ \text{Ho } \hat{S} \]

If this adjunction is an equivalence then \( S \) is a weak test category.

If this also holds true for all slices then \( S \) is a test category.

And if \( \int_S \cdot \gamma \text{ CAT} \) preserves products then \( S \) is a strict test category.

We can do synthetic homotopy theory in the category of presheaves for any (strict) test category \([\text{Gro}83]\).
Test Categories

Localization induces an adjunction on the homotopy categories.

\[
\begin{align*}
\mathcal{S} & \xleftarrow{y} \mathcal{S}/- \\
\mathcal{S} & \xrightarrow{\mathcal{S}} \mathcal{S}/- & \text{Ho } \mathcal{S} & \xrightarrow{\gamma} \text{Ho CAT} \\
\mathcal{S} & \xrightarrow{\mathcal{N}_\mathcal{S}} \text{Ho CAT} & \mathcal{S} & \xrightarrow{\gamma} \text{CAT} \\
\end{align*}
\]

If this adjunction is an equivalence then \(\mathcal{S}\) is a **weak test category**. If this also holds true for all slices then \(\mathcal{S}\) is a **test category**. And if \(\mathcal{N}_\mathcal{S}\cdot \gamma\) \(\text{CAT}\) preserves products then \(\mathcal{S}\) is a **strict test category**.
Test Categories

Localization induces an adjunction on the homotopy categories.

\[
\begin{align*}
\mathbb{S} & \xrightarrow{y} \mathbb{S}/- \\
\mathbb{S} & \xrightarrow{\int_S} \mathbb{S} \downarrow \mathbb{N}_S \\
\gamma \mathbb{S} & \xrightarrow{\gamma \mathbb{N}_S} \gamma \mathbb{C}AT \\
Ho \mathbb{S} & \xleftarrow{\gamma \mathbb{C}AT} Ho \mathbb{C}AT
\end{align*}
\]

If this adjunction is an equivalence then \( \mathbb{S} \) is a **weak test category**. If this also holds true for all slices then \( \mathbb{S} \) is a **test category**. And if \( \int_S \cdot \gamma \mathbb{C}AT \) preserves products then \( \mathbb{S} \) is a **strict test category**.

We can do synthetic homotopy theory in the category of presheaves for any (strict) test category [Gro83].
is a Strict Test Category

It suffices [Mal05; BM17] to observe that □ has finite products:

\[ 1 = [0] \quad \text{and} \quad [m] \times [n] = [m + n] \]

and an interval object:

\[ [0\mapsto 0], [0\mapsto 1] : \square ([0] \to [1]) \]

whose Yoneda image is *separated* (has the unique \( \hat{\square} (0 \to 1) \) as equalizer).
Test Functors

In the basic setup, we ask whether the slice functor induces an equivalence of homotopy categories.

For $\mathcal{S}$ a weak test category, $F$ is a weak test functor if:

1. $F(S)$ is aspheric (weakly equivalent to a point) for all $S : \mathcal{S}$,
2. the $\mathcal{S}$-nerve (right adjoint) preserves weak equivalences.

Any weak test functor induces an adjoint equivalence of homotopy categories. If all slices $\partial \mathcal{S} \cdot F : \mathcal{S}/S \to \mathcal{S} \to \text{Cat}$ are weak test functors then $F$ is a test functor.
Test Functors

In the basic setup, we ask whether the slice functor induces an equivalence of homotopy categories.

We can ask the same for an arbitrary functor $F : \mathcal{S} \to \text{CAT}$.
Test Functors
In the basic setup, we ask whether the slice functor induces an equivalence of homotopy categories.

We can ask the same for an arbitrary functor $F : \mathcal{S} \to \text{CAT}$.

For $\mathcal{S}$ a weak test category, $F$ is a weak test functor if:

1. $F(S)$ is aspheric (weakly equivalent to a point) for all $S : \mathcal{S}$,
2. the $\mathcal{S}$-nerve (right adjoint) preserves weak equivalences.

Any weak test functor induces an adjoint equivalence of homotopy categories.
Test Functors

In the basic setup, we ask whether the slice functor induces an equivalence of homotopy categories.

We can ask the same for an arbitrary functor $F : \mathbb{S} \to \text{CAT}$.

For $\mathbb{S}$ a weak test category, $F$ is a weak test functor if:

- $F(S)$ is aspheric (weakly equivalent to a point) for all $S : \mathbb{S}$,
- the $\mathbb{S}$-nerve (right adjoint) preserves weak equivalences.

Any weak test functor induces an adjoint equivalence of homotopy categories.

If all slices $\partial^- \cdot F : \mathbb{S}/S \to \mathbb{S} \to \text{CAT}$ are weak test functors then $F$ is a test functor.
It suffices [ZK12] to observe that □ is a full subcategory of $\mathbf{Cat}$ that:

- is closed under finite products,
- includes the walking interval,
- and excludes the walking nothing.
The category of presheaves for any test category can be equipped with a canonical *model structure* where \cite{cisinski}: 

cofibrations are the monomorphisms, 

weak equivalences are the maps that become weak equivalence in $\mathbf{CAT}$ under the category of elements functor.
Model Structure

The category of presheaves for any test category can be equipped with a canonical *model structure* where [Cis06]:

- **cofibrations** are the monomorphisms,

- **weak equivalences** are the maps that become weak equivalence in $\mathbf{CAT}$ under the category of elements functor.

Fibrant objects in this model structure on $\square$ have lots of fillings; e.g. from the “bent square” to the cube.

Implications??
Simplicial Cubes

There is a canonical functor $\square \to \hat{\Delta}$ mapping $[n] \mapsto (\Delta^1)^n$.

Since $\hat{\Delta}$ has pointwise products (i.e. $(X \times Y)f \simeq Xf \times Yf$), a simplex is degenerate in $X \times Y$ iff it is degenerate in $X$ and $Y$ simultaneously.
Simplicial Cubes

There is a canonical functor $\square \to \hat{\Delta}$ mapping $[n] \mapsto (\Delta^1)^\times n$. 

Since $\hat{\Delta}$ has pointwise products (i.e. $(X \times Y)f \cong Xf \times Yf$), a simplex is degenerate in $X \times Y$ iff it is degenerate in $X$ and $Y$ simultaneously.

Consider the nondegenerate $n$-simplices in $(\Delta^1)^\times n$.

Example: $n := 2$

$$([0, 1, 1], [0, 0, 1]) \quad \text{and} \quad ([0, 0, 1], [0, 1, 1])$$
Simplicial Cubes

There is a canonical functor $\square \to \hat{\Delta}$ mapping $[n] \mapsto (\Delta^1)^\times n$.

Since $\hat{\Delta}$ has pointwise products (i.e. $(X \times Y)f \cong Xf \times Yf$), a simplex is degenerate in $X \times Y$ iff it is degenerate in $X$ and $Y$ simultaneously.

Consider the nondegenerate $n$-simplices in $(\Delta^1)^\times n$.

Example: $n := 2$

$$([0, 1, 1] , [0, 0, 1]) \quad \text{and} \quad ([0, 0, 1] , [0, 1, 1])$$

Zipping these:

$$[(0 , 0), (1 , 0), (1 , 1)] \quad \text{and} \quad [(0 , 0), (0 , 1), (1 , 1)]$$
Simplicial Cubes

There is a canonical functor \( \square \to \widehat{\Delta} \) mapping \([n] \mapsto (\Delta^1)^\times n\).

Since \( \widehat{\Delta} \) has pointwise products (i.e. \((X \times Y)f \cong Xf \times Yf\)), a simplex is degenerate in \(X \times Y\) iff it is degenerate in \(X\) and \(Y\) simultaneously.

Consider the nondegenerate \(n\)-simplices in \((\Delta^1)^\times n\).

Example: \(n := 2\)

\[
([0, 1, 1], [0, 0, 1]) \quad \text{and} \quad ([0, 0, 1], [0, 1, 1])
\]

Zipping these:

\[
[(0, 0), (1, 0), (1, 1)] \quad \text{and} \quad [(0, 0), (0, 1), (1, 1)]
\]

We recover the \emph{triangulation} profunctor \(t : \square \to \Delta\).
Triangulating Cubical Sets

Since □ is small and \( \hat{\Delta} \) is cocomplete we can extend triangulation along Yoneda:

\[
\begin{array}{ccc}
\square & \xrightarrow{y} & \triangleleft \\
\downarrow^{t!} & & \downarrow^{t} \\
\square & \xrightarrow{t} & \hat{\Delta}
\end{array}
\]

which lets us triangulate cubical sets.
Since $\square$ is small and $\hat{\Delta}$ is cocomplete we can extend triangulation along Yoneda:

\[
\begin{array}{ccc}
\square & \xrightarrow{y} & t \\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu
Summary

The ordered cubes are a shape category with good combinatorial and homotopical properties.

They may also provide an interesting foundation for a cubical type theory.

I am grateful to several workshop participants for pointing out to me related work of which I was unaware. In particular, I would like to acknowledge a recent preprint by Chris Kapulkin containing joint work done with Vladimir Voevodsky, which contains many of the results discussed here – and much more besides:


Marco Grandis and Luca Mauri. “Cubical Sets and their Site”. In: Theory and Application of Categories 11.8 (2003), pp. 185–211.


