Varieties of Cubical Sets

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Summary

We study a variety of notions of *cubical sets*
based on *substructural algebraic theories*
presenting *monoidal categories*.

We explore the proof theory and homotopy theory of these cubical sets:
we determine which are canonical for their language,
and which are (strict) *test categories* in the sense of Grothendieck.

Monoidal Algebraic Theories

Structural Rules

Substructural languages let us restrict how context variables may appear in terms.

We consider the following set of **structural rules**:

- **weakening** (w) allows unused variables: $x\,,y \vdash t(x)$
- **exchange** (e) allows variable order permutation: $x\,,y \vdash t(y\,,x)$
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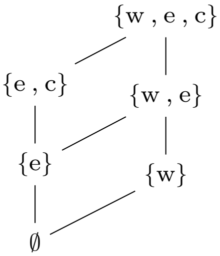
and its subset lattice where $\mathrm{c} \Rightarrow \mathrm{e}$:



(1)

Interpreting Structural Rules

Interpretations for our languages will be in a **monoidal category** $(\mathcal{E}, \otimes, 1)$ with a single generating object $\mathrm{X} : \mathcal{E}$.

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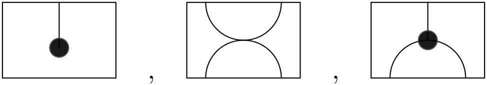

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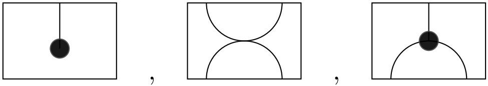

When $\mathcal{E}$ is *symmetric monoidal*, τ is the braiding.
When $\mathcal{E}$ is *cartesian monoidal*, ε the unique map to 1 and δ is the diagonal.

Algebraic Signatures

Our languages are all single-sorted and algebraic (co-arity one).

We consider the following set of **function symbols**:

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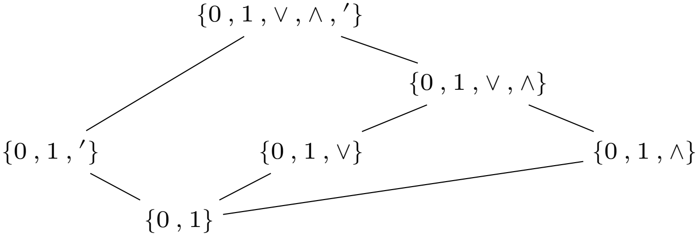

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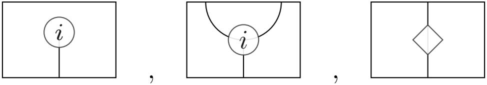

Cubical Theories

Definition (cubical language)

Let $\mathrm{L}_{(a,b)}$ be the language with *structural rules* $a \subseteq$ "wec" allowed by (1) and *signature* $b \subseteq$ "$\vee \wedge\ '$" allowed by (2) (with 0 and 1 assumed).

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Definition (syntactic category of a cubical theory)

For T an equational theory in a cubical language $\mathrm{L}_{(a,b)}$,
let $\mathbb{C}_{(a,b)}(\mathrm{T})$ be the **syntactic category** of T, with:

- morphisms generated by a and b,
- morphism equality determined by T.

Standard Structures

- standard **topological interval**: $\mathbb{I} := [0\,,1]$ in $\textsc{Top}$
  with $x \vee y = \max(x\,,y)$, $x \wedge y = \min(x\,,y)$, $x' = 1 - x$;

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- standard **two-element set**: $\mathbb{2} := \{0\,,1\}$ in $\textsc{Set}$
  with the relations above;
- the **three-element Kleene algebra**: $\mathbb{3} := \{0\,,u\,,1\}$ with $u' = u$;
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$\mathbb{2}$ gives the theory of boolean algebras,
$\mathbb{3}$ gives the theory of Kleene algebras and
$\mathbb{D}$ gives the theory of de Morgan algebras. [GWW03]

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For cubical language $\mathrm{L}_{(a,b)} \lneqq \mathrm{L}_{(\mathrm{wec},\wedge\vee')}$ we have $\mathrm{Th}(\mathbb{I}) = \mathrm{Th}(\mathbb{2})$.
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For $\mathrm{L}_{(\mathrm{wec},\wedge\vee')}$ we have $\mathrm{Th}(\mathbb{I}) = \mathrm{Th}(\mathbb{3})$. [GWW03]

Corollary 1.3

Each of our canonical cube categories has decidable morphism equality.

Cubical Axioms

Axiom	Lang. req.	Name
$x \vee (y \vee z) = (x \vee y) \vee z$	$(\cdot\,, \vee)$	$\vee$-associativity
$0 \vee x = x = x \vee 0$	$(\cdot\,, \vee)$	$\vee$-unit
$1 \vee x = 1 = x \vee 1$	$(\mathrm{w}\,, \vee)$	$\vee$-absorption
$x \vee y = y \vee x$	$(\mathrm{e}\,, \vee)$	$\vee$-symmetry
$x \vee x = x$	$(\mathrm{ec}\,, \vee)$	$\vee$-idempotence
$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	$(\cdot\,, \wedge)$	$\wedge$-associativity
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$x'' = x$	$(\cdot\,, ')$	$'$-involution
$0' = 1$	$(\cdot\,, ')$	$'$-computation
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	$(\mathrm{ec}\,, \vee\wedge)$	distributive law 1
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	$(\mathrm{ec}\,, \vee\wedge)$	distributive law 2
$x = x \vee (x \wedge y) = x \wedge (x \vee y)$	$(\mathrm{wec}\,, \vee\wedge)$	lattice-absorption
$(x \vee y)' = x' \wedge y'$	$(\cdot\,, \vee\wedge')$	de Morgan's law
$x \wedge x' \leq y \vee y'$	$(\mathrm{wec}\,, \vee\wedge')$	Kleene's law

Cubical Axiomatizations

The theory of each canonical cube category with weakening is axiomatized by the equations expressible in the corresponding language.

For $\mathrm{L}_{(\mathrm{wec},\wedge\vee')}$ we also have the non-canonical cube categories for :

- de Morgan algebras, $\mathbb{C}_{\mathrm{dM}}$, satisfying all axioms except Kleene's law, (notable for being the basis of the type theory for a programming language [Coh+15])
- boolean algebras, $\mathbb{C}_{\mathrm{BA}}$, additionally satisfying *excluded middle*: $x \vee x' = 1$.

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For each cube category $\mathbb{C}$, we write "$[n]$" for $\mathrm{X}^{\otimes n}$ ($= [\![x_1 , \cdots , x_n]\!]$).

n-Dimensional Cubes

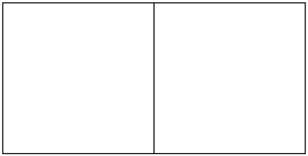
0-dimensional cube (point):

$$[0] \quad = [\![\cdot]\!]$$

n-Dimensional Cubes

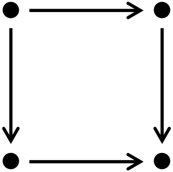
1-dimensional cube (interval):

$$[1] \quad = [\![x]\!]$$

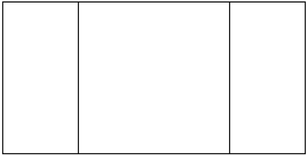


n-Dimensional Cubes

2-dimensional cube (square):



$$[2] \quad = [\![x \ , \ y]\!]$$



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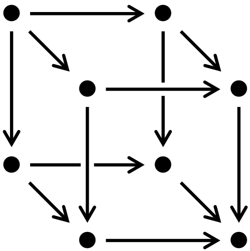
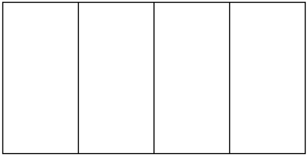
3-dimensional cube (cube): 

$$[3] \quad = [\![x \, , \, y \, , \, z]\!]$$



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n-dimensional cube:

???

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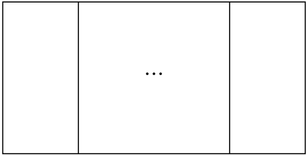


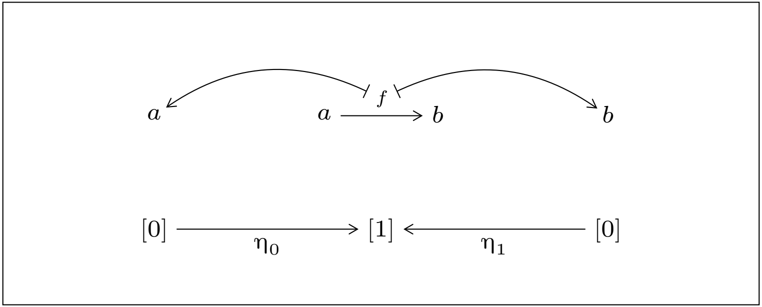
Cubical Sets

A **cubical set** is a presheaf on a cube category (i.e. a functor $\mathrm{X} : \mathbb{C}^{\circ} \to \textsc{Set}$):

- an object $[n] : \mathbb{C}$ determines a set of n-cubes,
- an arrow $\varphi : \mathbb{C}\left([n] \to [m]\right)$ determines a function $\mathrm{X}(\varphi)$ from m-cubes to n-cubes.

Cube Faces

In the canonical cube category $\mathbb{C}_{(\cdot,\cdot)}$, the η_i generate the **face maps**:



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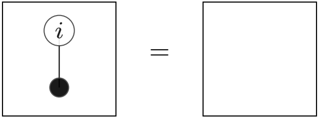
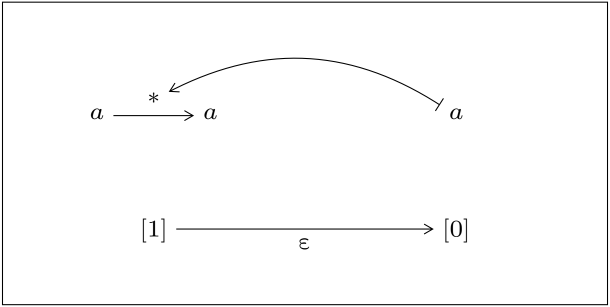
This gives the **face-degeneracy laws**:





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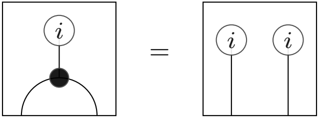
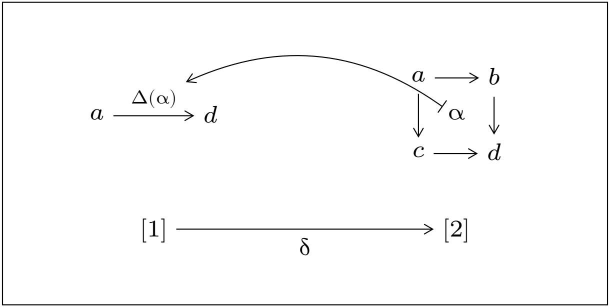
and δ interacts with the η_i by the **face-diagonal laws**:





Cubes with Reversals

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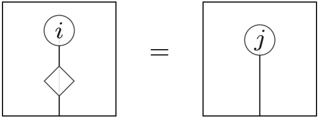
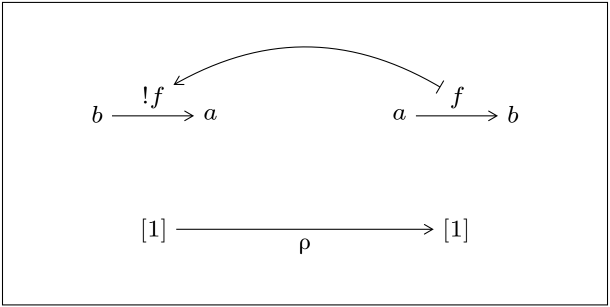


where $i \neq j$



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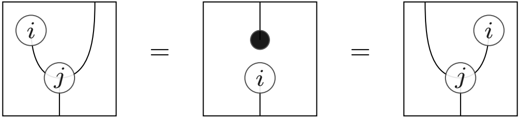
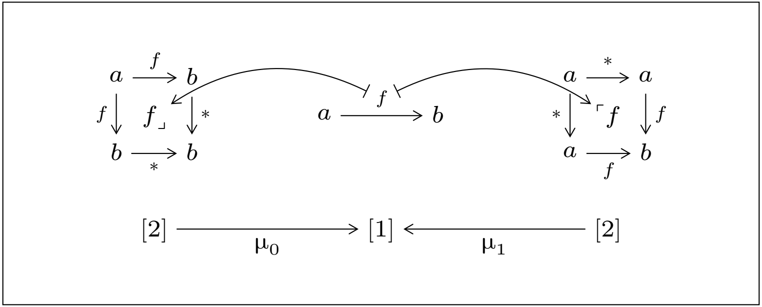
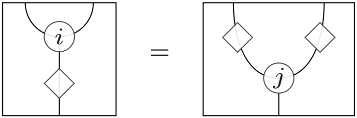
Each η_i is an absorbing element for μ_j ($i \neq j$), giving the **dioid laws** [GM03]:



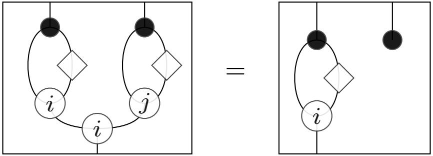


The Full Signature

In the canonical cube category $\mathbb{C}_{(\cdot,\vee\wedge')}$, reversal interacts with connections by the **de Morgan law**:



And in the canonical cube category $\mathbb{C}_{(\mathrm{wec},\vee\wedge')}$, by the algebraic characterization of order in a lattice, we have the **Kleene law**:



Homotopy of Cubes

Classical Homotopy

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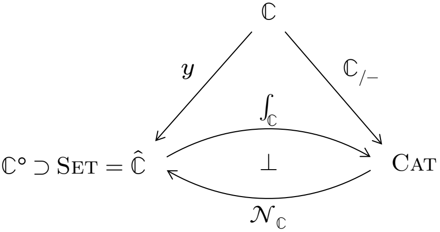
The classical **homotopy category** is formed by formally inverting (localizing at) the (weak) homotopy equivalences:

$$\textsc{Top} \longmapsto \textsc{Hot} = \textsc{Ho}(\textsc{Top}) = \textsc{Top}[\mathcal{W}^{-1}]$$

Synthetic Homotopy

For any small category $\mathbb{C}$, the slice functor, $\mathbb{C}_{/-} : \mathbb{C} \to \textsc{Cat}$ uniquely determines an adjunction:



where $\int_{\mathbb{C}}$ gives the **category of elements** of a presheaf,
and $\mathrm{N}_{\mathbb{C}}$ is the **nerve** functor: $\mathcal{N}_{\mathbb{C}}(\mathbb{D})(\mathrm{A}) = \textsc{Cat}\,(\mathbb{C}_{/\mathrm{A}} \to \mathbb{D})$.

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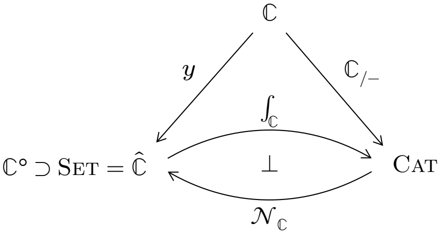

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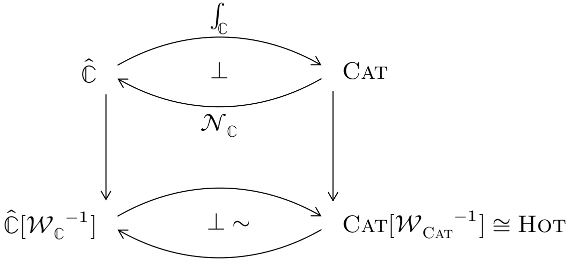
Via simplicial sets, the category $\mathrm{C{\small AT}}$ also presents the homotopy category $\mathrm{H{\small OT}}$.

Grothendieck showed this permits the study of **synthetic homotopy** for the category of presheaves over any small category. [Gro83]

Test Categories

- $\mathbb{C}$ is a **weak test category** if the adjunction extends to an adjoint equivalence on the localizations:



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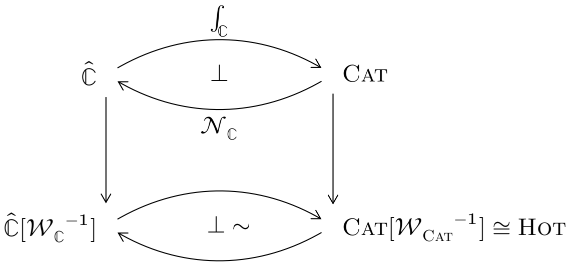

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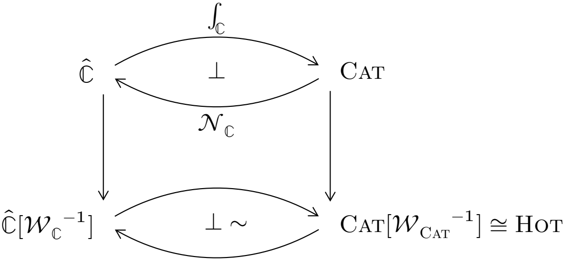

- $\mathbb{C}$ is a **test category** if it, and each of its slices, is weak test.
- $\mathbb{C}$ is a **strict test category** it is test and the functor $\hat{\mathbb{C}} \to \mathrm{HOT}$ preserves finite products.

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$$
\begin{array}{ccc}
\hat{\mathbb{C}} & \underset{\mathcal{N}_{\mathbb{C}}}{\overset{\int_{\mathbb{C}}}{\rightleftarrows}} \;\bot & \mathrm{CAT} \\
\downarrow & & \downarrow \\
\hat{\mathbb{C}}[\mathcal{W}_{\mathbb{C}}{}^{-1}] & \rightleftarrows \;\bot\; \sim & \mathrm{CAT}[\mathcal{W}_{\mathrm{CAT}}{}^{-1}] \cong \mathrm{HOT}
\end{array}
$$

- $\mathbb{C}$ is a **test category** if it, and each of its slices, is weak test.
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Idea: presheaves of test categories have “the right homotopy”, which is preserved under products for strict test categories.

Canonical Cube Test Categories

Theorem 3.1

- The canonical cube categories for theories with the structural rule of weakening are test categories.

a\b	$\cdot$	$'$	$\vee$	$\wedge$	$\vee\wedge$	$\vee\wedge'$
w	t	t	t	t	t	t
we	t	t	t	t	t	t
wec	t	t	t	t	t	t

Which canonical cube categories $\mathbb{C}_{(a,b)}$ are test (t) or even strict test (st).

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- The canonical cube categories for theories with contraction as well are strict test categories.

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a\b	$\cdot$	$'$	$\vee$	$\wedge$	$\vee\wedge$	$\vee\wedge'$
w	t	t	st	st	st	st
we	t	t	st	st	st	st
wec	st	st	st	st	st	st/st/st

The bottom-right corner refers to the cube categories for de Morgan, Kleene and boolean algebras.

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w	t	t	st	st	st	st
we	t	t	st	st	st	st
wec	st	st	st	st	st	st/st/st

Upshot: having either the diagonal or a connection suffices for strict test.

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