A higher-dimensional type theory depends on a notion of higher-dimensional abstract spaces.

Many choices: globular, simplicial, cubical, opotopic, etc.

We want abstract spaces with good topological properties as well as good combinatorial and computational properties.

Lately, we have been thinking about cubical structure.
Several cubical structures have been proposed as a basis for models of higher-dimensional type theory.

We survey some of their features.
Abstract Cubes

A cube category is a symmetric monoidal category with a distinguished object, the abstract interval, $I$.

In a cube category, $\square$, for each $n \in \mathbb{N}$, we have an abstract $n$-dimensional cube, $[n] := I \otimes \cdots \otimes I$. 
0-Dimensional Cube (point)
1-Dimensional Cube (interval)
2-Dimensional Cube (square)
3-Dimensional Cube (cube)
$n$-Dimensional Cube

[Diagram of an $n$-dimensional cube with ellipses indicating higher dimensions]

$[n]$
Cubiness

We seek an equational presentation of cubes so we can describe cubes of any dimension and the relationships between them.
A cubical set is a presheaf on a cube category:

\[
\begin{array}{llll}
a & b & \rightarrow f & \rightarrow A \\
\downarrow & \downarrow & \downarrow & \downarrow \\
d & c & \rightarrow g & \rightarrow A \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\rightarrow h & \rightarrow & B & \\
\end{array}
\]

The cubes we are interested in reside in the fibers, sorted by dimension.

Maps between abstract cubes determine contravariant functions describing relationships between cubes.
Boundary Maps

An abstract interval has two distinguishable boundary points. This gives us a notion of a path.

\[ \partial^-, \partial^+ : \square ([0] \to [1]) \]

\[ a \leftarrow b \quad \xrightarrow{f} \quad a \rightarrow b \]

\[ [0] \quad \xrightarrow{\partial^-} \quad [1] \]
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![Diagram showing a path between two points labeled a and b, with arrows indicating direction and a boundary map from [0] to [1].]
Degeneracies

Represent the idea of a trivial path:

\[ \varepsilon : \square ([1] \rightarrow [0]) \]
Boundary-Degeneracy Laws

\[ \partial^i \cdot \varepsilon = \text{id}(0) \]
\( \Box(\partial, \varepsilon) \)

**Generators**

\[ \partial^- , \partial^+ , \varepsilon \]

**Relations**

\[ \partial^- \varepsilon = \varepsilon = \partial^+ \varepsilon \]
Diagonal Maps

Represent the idea of a path cutting through the middle of a square:

$$\Delta : \square ([1] \rightarrow [2])$$
So far, the diagonal is under-specified: we don’t say how to cut through the middle of a square.

But there is still something that we know for certain: its boundary.

generator

relations
Symmetrical Diagonals

If the diagonal cuts through the square in “a straight line” then we get more laws:

**diagonal-diagonal law**

\[ \Delta \cdot (\Delta \otimes [1]) = \Delta \cdot ([1] \otimes \Delta) \]

represents cutting through the middle of a 3-cube.
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Symmetrical Diagonals

Also, putting the interval in the diagonal of the square and then squishing the square back into the interval along either dimension is identity:

**diagonal-degeneracy laws**

\[
\Delta \cdot (\varepsilon \otimes [1]) = \text{id}([1]) = \Delta \cdot ([1] \otimes \varepsilon)
\]
You may recognize these as the *comonoid* laws:

If we extend this comonoid structure *naturally* to all $[n]$, then the monoidal structure becomes *cartesian*.
Cartesian Cubical Sets

Cartesian cubical sets have several good properties, eg:

- It is a \textit{strict test category} (has the “right homotopy theory”).
- Contexts of dimension variables behave \textit{structurally}
  (admit exchange, weakening and contraction).
Reversals

Represent the idea of following a path *backwards*:

\[ \rho : \square (\{1\} \rightarrow \{1\}) \]
\[ \square(\partial, \varepsilon, \rho) \]

The theory \( \square(\partial, \varepsilon) \) plus:

**generator**

\[
\rho^2 = \rho, \\
\rho \varepsilon = \varepsilon, \\
\partial^- \rho = \partial^+ \\
\partial^+ \rho = \partial^- 
\]

**relations**
Connections

Represent another kind of degeneracy, identifying *adjacent*, rather than *opposite*, sides of an abstract cube. They collapse a square to an interval, like a folding paper fan:

\[ \gamma : \square ([2] \rightarrow [1]) \]
The theory $\square(\partial, \varepsilon, \gamma)$ plus:

generator

relations

$(\gamma, \partial^+)$ forms a monoid:
\( \Box(\partial, \varepsilon, \gamma) \)

The theory \( \Box(\partial, \varepsilon) \) plus:

- generator \( \gamma \)

relations

\( \partial^- \) is an *absorbing element* (zero) for this monoid:
\[ \square(\partial, \varepsilon, \gamma) \]

The theory \( \square(\partial, \varepsilon) \) plus:

- **generator**

- **relations**

\( \varepsilon \) is a *morphism* for this monoid structure:

\[ \varepsilon \varepsilon = \varepsilon \varepsilon \]  

(plus boundary-degeneracy law from before)
Connections and Reversals

Using reversal, we get three more connections, one for “folding” the square at each of its corners:

\[
\begin{array}{c}
\begin{array}{ccc}
\gamma^{++}(f) & f & \gamma^{--}(f) \\
\downarrow & \downarrow & \downarrow \\
\gamma^{-+}(f) & f & \gamma^{--}(f) \\
\downarrow & \downarrow & \downarrow \\
\gamma^{+-}(f) & f & \gamma^{++}(f) \\
\downarrow & \downarrow & \downarrow \\
\gamma^{--}(f) & f & \gamma^{++}(f) \\
\downarrow & \downarrow & \downarrow \\
\gamma^{-+}(f) & f & \gamma^{--}(f) \\
\downarrow & \downarrow & \downarrow \\
\gamma^{+-}(f) & f & \gamma^{++}(f) \\
\downarrow & \downarrow & \downarrow \\
\gamma^{--}(f) & f & \gamma^{++}(f) \\
\end{array}
\end{array}
\]
Composition

Fancier structures, such as **cubical groupoids**, extend cubical sets with a composition structure.

E.g. $f +_x g$:

$$
\begin{array}{c}
a \\ b \\ d
\end{array}
\begin{array}{c}
f \\ g
\end{array}
\begin{array}{c}
b \\ c \\ e
\end{array}

E.g. $A +_x B$:

$$
\begin{array}{c}
a \\ d
\end{array}
\begin{array}{c}
A \\ B
\end{array}
\begin{array}{c}
b \\ e \\ c
\end{array}

(laws available but elided)
In some cases, we may be able to *subdivide* cubes in a canonical way.

E.g. *padding*:

\[
a \overset{*}{\rightarrow} a \overset{f}{\rightarrow} b \overset{*}{\rightarrow} b
\]
Cubical sets with the box-filling property are called “Kan”:

\[
\forall \quad f \downarrow \quad g \quad \exists \quad f \downarrow \quad A \quad \downarrow h
\]

Kan cubical sets that are uniform with respect to degeneracies are important for interpreting higher-dimensional type theories.
Constructive Box Filling

With reversals and connections, we can constructively fill padded boxes in a cubical set.

A right adjoint to subdivision then lets us fill boxes in the fibrant replacement of a cubical set.
Thanks!