Categorical Bottom-Up Semantics for Logic Programming

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Outline

The Dream of Programming in Logic

Why try to program in logic?

- Logic is declarative: say *what* you want, not *how* to get it
- Transfer work from people to machines
- Reduce error
- Reduce the "semantic gap" between specifications and programs

To a limited extent, the dream has been realized with definite clause logic.

The Dream Deferred

Then why are we not satisfied?

- Systems lack many basic programming features (e.g. modules, abstract datatypes, higher-order functions). Even arithmetic can be cumbersome.
- Some features get bolted on but are often not consistent with the the underlying logical theory.
- Still a large semantic gap (sometimes made worse by the shortcomings of the systems).

What can we do?

- Extend the logic
- Add non-logical features in a principled way that doesn't destroy declarative transparency of programs

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2 definite clause logic programming in SET

- Definite Clause Logic
- Herbrand Models
- Immediate Consequences
- Answers and Unification
- SLD Resolution

Definite Programs and Goals

definite goal and program clauses Let $\mathcal{L}$ be a first-order language. We define the set of **goal formulas** $\mathcal{G}$ and **program formulas** $\mathcal{D}$ (also called definite clauses) by the following grammar over atoms, A:

$$G ::= \top \mid A \mid G \wedge G$$
$$D ::= A \leftarrow G$$

definite program A finite $P \subset \mathcal{D}$

The World According to Herbrand

Herbrand universe ($U_{\mathcal{L}}$) For language $\mathcal{L}$, the set of all closed *terms* that can be formed from the constant and function symbols of $\mathcal{L}$.

Herbrand base ($B_{\mathcal{L}}$) For language $\mathcal{L}$, the set of all closed *atomic formulas* (atoms) that can be formed from the predicate symbols of $\mathcal{L}$ with elements of the Herbrand universe as arguments.

We can relativize these a set of $\mathcal{L}$-formulas P, by writing U_P and B_P

Herbrand Interpretations and Models

Herbrand interpretation Any subset of the Herbrand base (representing the closed atoms that are true in the interpretation). The set of all such is written $2^{B_{\mathcal{L}}}$.

Herbrand model for a set of closed $\mathcal{L}$-formulas T, a Herbrand interpretation of $\mathcal{L}$ that is a model for T.

Definite Clauses and Herbrand Models

proposition If a set of *definite clauses* is satisfiable then it has a Herbrand model.

model intersection property If P is a definite program and $\{M_i\}_{i:I}$ a non-empty set of Herbrand models for P, then $\bigcap_{i:I} M_i$ is also a Herbrand model for P.

least Herbrand model (M_P) The intersection of all Herbrand models for P. (this always exists since $B_P \models P$ so the set of such is nonempty)

theorem (Kowalski & van Emden 1976)

$$M_P = \{A : B_P \mid P \models A\}$$

The T_P Operator on Herbrand Interpretations

immediate consequences operator (T_P) For definite program P, the function on Herbrand interpretations

$$T_P : 2^{B_P} \to 2^{B_P}$$

defined by

$$I \longmapsto \{A : B_P \mid (A \leftarrow A_1 \wedge \cdots \wedge A_n) \text{ is a ground instance of of a clause in } P \text{ and } \{A_1, ..., A_n\} \subset I\}$$

Least Herbrand Model as a Fixpoint

proposition For Herbrand interpretation I of definite program P,

$$I \models P \iff T_P(I) \subset I$$

fixed point characterization of least Herbrand model For definite program P,

$$\begin{aligned} M_P &= \text{lfp}\ (T_P) \\ &= T_P \uparrow (\emptyset) \\ &= T_P(\emptyset) \cup T_P(T_P(\emptyset)) \cup \cdots \end{aligned}$$

Correct Answers

correct answer For definite program P and definite goal G, a substitution θ such that $P \models \forall(G\,\theta)$

proposition For definite program P, definite goal G, and substitution θ such that $G\,\theta$ is ground, θ is a correct answer iff $M_P \models G\,\theta$.

Unification

unifying substitution (unifier) For expressions E and F, a substitution θ such that $E\,\theta = F\,\theta$.

most general unifier (mgu) Unifier θ is most general if for every unifier φ, there is a substitution σ such that $\varphi = \theta \cdot \sigma$

unification theorem Unification in first order logic is decidable. If a pair of terms are unifiable, then the unification algorithm computes their mgu, else it reports failure.

SLD Derivation

SLD step ($\rightsquigarrow$) For definite program clause $P = A \leftarrow B_1 \wedge \cdots \wedge B_m$, definite goal clause $G = A_1 \wedge \cdots \wedge A_i \wedge \cdots \wedge A_n$ and substitution $\theta = mgu(A_i, A)$,

$$G \underset{P,A_i}{\overset{\theta}{\rightsquigarrow}} G'$$

where

$$G' = (A_1 \wedge \cdots \wedge A_{i-1} \wedge \overbrace{B_1 \wedge \cdots \wedge B_m}^{\text{tail of P replaces } A_i} \wedge A_{i+1} \wedge \cdots \wedge A_n)\ \theta$$

this is essentially tracing the implications, "$\leftarrow$" from the program backwards.
i.e. "$G\ \theta$ would be true if $G'\ \theta$ were true"

SLD Derivation

SLD derivation For definite program P and goal G, a (possibly infinite) chain of derivation steps starting from $G = G_0$:

$$G_0 \overset{\theta_0}{\underset{P_0,A_0}{\rightsquigarrow}} G_1 \overset{\theta_1}{\underset{P_1,A_1}{\rightsquigarrow}} \cdots$$

where A_i is an atom in G_i and P_i is a clause in P (with variables renamed to avoid clashes).

SLD proof A finite SLD derivation ending in $\top$ (success).

computed answer The substitution that is the composition of mgus in a proof. (this is what is returned by a PROLOG interpreter)

Soundness and Completeness of SLD Derivation

soundness of SLD resolution (Clark 1979) For definite program P and definite goal G, every computed answer for $(P \vdash_{SLD} G)$ is a correct answer.

completeness of SLD resolution (Clark 1979) For definite program P and definite goal G, for every correct answer φ for (P, G) there exists a computed answer θ and substitution γ such that $\varphi = \theta \cdot \gamma$.

3 categorical syntax, derivation and semantics for definite clause logic

- categorical syntax
- categorical SLD derivation
- categorical semantics

Categorical Signature

a triple, $\Sigma = (\mathbb{C}, \mathcal{D}, \vec{B})$ where:

- $\mathbb{C}$ is a category with finite products and distinguished limits
  The objects of $\mathbb{C}$ are called **sorts**
- $\mathcal{D} \subset \mathbb{C}^{\rightarrow}$
  whose elements are called **terms**
- $\vec{B} = B_1, ..., B_n$ is a sequence of distinct monics in $\mathbb{C}$ with the property that their pullbacks along coterminal arrows in $\mathcal{D}$ exist.
  We call these **predicate tokens**

Interpretation of a Language in a Signature

For sorted first-order language $\mathcal{L}$ and signature $\Sigma = (\mathbb{C}, \mathcal{D}, \vec{B})$, a **syntactic interpretation** of $\mathcal{L}$ into Σ is a function $M : \mathcal{L} \to \mathbb{C}$ such that:

sort	σ	$\longmapsto$	$M(\sigma)$ where $\mathrm{id}_\sigma \in \mathcal{D}$	
constant symbol	$c : \sigma$	$\longmapsto$	$M(c) : \mathbf{1} \to M(\sigma)$	
function symbol	$f : (\rho \to \sigma)$	$\longmapsto$	$M(f) : (M(\rho) \to M(\sigma))$	$\in \mathcal{D}$
relation symbol	$R : \sigma$	$\longmapsto$	$M(R) : (\rho \rightarrowtail M(\sigma))$	$\in \vec{B}$

such that the image of every term in $\mathcal{L}$ (inductively generated by the function symbols) is a term in $\mathbb{C}$ – i.e. in $\mathcal{D}$.

Syntactic Interpretation of Variables

Categorical syntax is inherently variable-free, but to keep track of which variables in the language are under consideration, we use the following convention:
For $\vec{x}$ a sequence of free $\mathcal{L}$-variables of sort $\vec{\rho}$:

$$M(\vec{x}) = M(\rho_1) \times ... \times M(\rho_n) : \mathbb{C}$$

Syntactic Interpretation of Terms

For term t of (out) sort ρ with free variables among $\vec{x}$:

$$M_{\vec{x}}(t) : \mathbb{C}(M(\vec{x}) \to M(\rho))$$

is defined:

variable $M_{\vec{x}}(x_i) = \pi_i$, a canonical projection

constant symbol $M_{\vec{x}}(c) = \ !_{M(\vec{x})} \cdot M(c)$

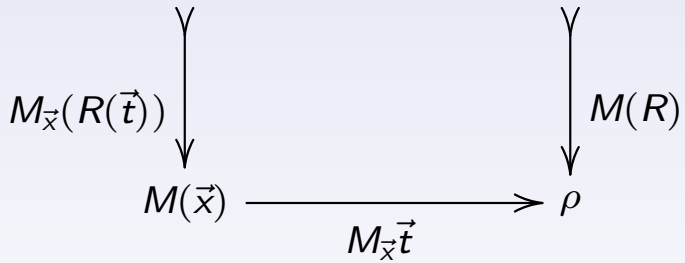
function symbol applied to terms $M_{\vec{x}}(f(\vec{t})) = \langle M_{\vec{x}}(t_1), ..., M_{\vec{x}}(t_n) \rangle \cdot M(f)$

Syntactic Interpretation of Atomic Formulas

For predicate token R of sort ρ and terms $\vec{t}$ with their free variables among $\vec{x}$

$$M_{\vec{x}}(R(\vec{t}))$$

is the monic targeted at $M(\vec{x})$ that is the *pullback* of R along $M_{\vec{x}}\vec{t}$ $\quad (= \langle M_{\vec{x}}(t_1), ..., M_{\vec{x}}(t_n) \rangle)$



(recall that the pullback of a monic is monic)

Definite Clause Diagrams

For a definite program or goal clause C with its free variables among $\vec{x}$, the set of monics interpreting its atomic formulas, each targeted at sort $M(\vec{x})$. For program clause diagrams, we also annotate which monic interprets the head of the clause.

note For definite clause logic, we can get away with such a simple notion of formula diagram because goals are just conjunctions of atoms.

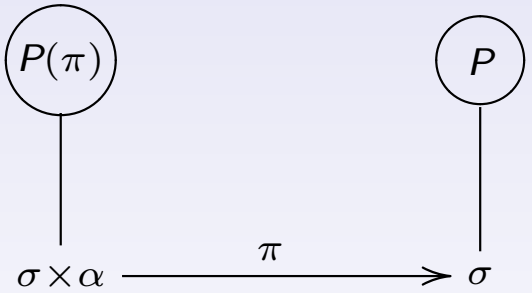
The empty diagram corresponds to goal "$\top$".

Sort Extension

Just as we can add "dummy variables" to a formula of first order logic:

$$\varphi(x) \longmapsto \varphi(x, y)$$

we can extend the sort of a diagram by pulling it back along a projection:



This allows us to bring any set of diagrams over a common sort, namely that of a product of their individual sorts.

Categorical Substitution

Let $\theta = \{\vec{x} := \vec{t}\}$ be an $\mathcal{L}$-substitution and $\vec{y}$ be a sequence of distinct variables containing all free variables in $\vec{t}$. Then the categorical substitution $\Theta_{\vec{y}}$ is the morphism:

$$\Theta_{\vec{y}} = \langle M_{\vec{y}}(t_1), ..., M_{\vec{y}}(t_n) \rangle : M(\vec{y}) \longrightarrow M(\vec{x})$$

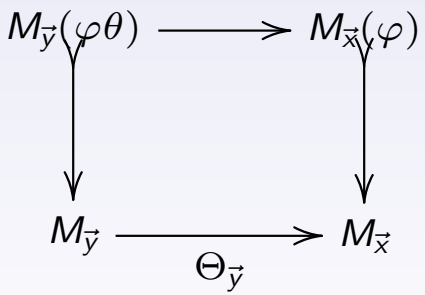
- The contravariance is due to the fact that substitution acts on inputs (the free variables) rather than outputs.
- since they're just arrows, substitutions act on terms and formulas the same way other arrows do.

Categorical Substitution

applying substitutions to terms For $\mathcal{L}$-term s with free variables among $\vec{x}$ and substitution θ as above:

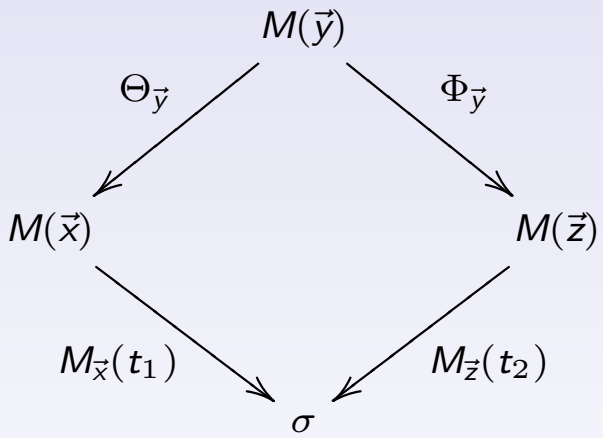
$$M_{\vec{y}}(s\,\theta) = \Theta_{\vec{y}} \cdot M_{\vec{x}}(s)$$

applying substitutions to atomic formulas For atomic formula φ, $M_{\vec{y}}(\varphi\theta) = (\Theta_{\vec{y}})^{\#}(M_{\vec{x}}(\varphi))$:



Categorical Unification

unification of terms terms t_1, t_2 are unifiable if there exist categorical substitutions $\Theta_{\vec{y}}, \Phi_{\vec{y}}$ such that



Categorical Unification

unification of atomic formulas atomic formulas $P_1(\vec{s}), P_2(\vec{t})$ are unifiable if

- the interpretations of their predicate symbols are identical
- the interpretations of their argument terms are unifiable

most general unifiers in either case, a unifier is most general if the square is a pullback.

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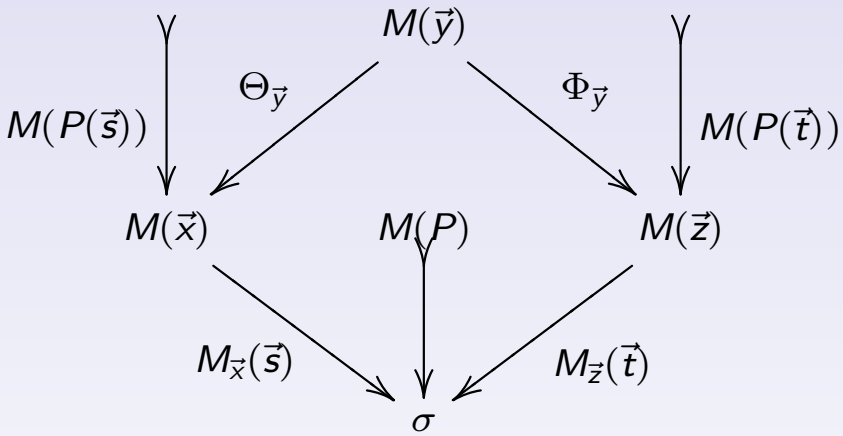


most general unifiers in either case, a unifier is most general if the square is a pullback.

Categorical SLD Derivation Step

For goal clause diagram $M_{\vec{x}}(G = A_1 \wedge \cdots \wedge A_n)$ and program clause diagram $M_{\vec{z}}(C = A \leftarrow B_1 \wedge \cdots \wedge B_m)$, goal clause diagram $M_{\vec{y}}(G')$ is derived from $M_{\vec{x}}(G)$ and $M_{\vec{z}}(C)$ by $(\Theta_{\vec{y}}, \Phi_{\vec{y}})$ if

- $M_{\vec{x}}(A_i)$ is the selected atom of $M_{\vec{x}}(G)$
- $(\Theta_{\vec{y}}, \Phi_{\vec{y}})$ is the mgu of $M_{\vec{x}}(A_i)$ and $M_{\vec{z}}(A)$

in which case,

$$\begin{array}{rlll} M_{\vec{y}}(G') = & \Theta_{\vec{y}}{}^{\sharp}\ (M_{\vec{x}}(A_1) & \cup \cdots \cup & M_{\vec{x}}(A_{i-1})) \\ \cup & \Phi_{\vec{y}}{}^{\sharp}\ (M_{\vec{z}}(B_1) & \cup \cdots \cup & M_{\vec{z}}(B_m)) \\ \cup & \Theta_{\vec{y}}{}^{\sharp}\ (M_{\vec{x}}(A_{i+1}) & \cup \cdots \cup & M_{\vec{x}}(A_n)) \end{array}$$

where $\Theta_{\vec{y}}$ is the **goal substitution** and $\Phi_{\vec{y}}$ the **program substitution**.

Categorical SLD Derivations and Answers

- We write $M_{\vec{x}}(G) \overset{\Theta_{\vec{y}}, \Phi_{\vec{y}}}{\underset{A_i, C}{\rightsquigarrow}} M_{\vec{y}}(G')$ for a derivation step.

  (we suppress the program substitution if it is not relevant)
- By chaining these steps, we get a **categorical SDL derivation**.
- If the derivation is finite and the last goal is the empty goal diagram over its sort, we have a **categorical SLD proof**.
- The **computed answer** of a proof is the composition of the *goal substitutions*

$$\Theta_n \cdot \Theta_{n-1} \cdot \ldots \cdot \Theta_1$$

  (note the reversal of order, corresponding to the contravariance of categorical substitution)

Categorical Semantics

- Once we are able to translate a language into a category that we will call **syntactic**, we want to be able to do model theory in the context of categories as well.
- Structure-preserving (in this case, limit-preserving) functors allow us to interpret one logic programming category in another.
- We call the target of a categorical interpretation the **semantic** category.

4 Yoneda semantics for a category of generic predicates

- syntactic categories with generic predicates
- interpretation in the Yoneda topos
- consequences

Generic Predicates

In pure logic programming, we want to begin with a syntactic category that is as free as possible, and for the logical structure to be supplied (entirely) by a program.

generic predicate We say that monic X is generic with respect to a collection of arrows, $\mathcal{D}$ if:

- $\forall t \in \mathcal{D}$ coterminal with X the pullback $t^{\#}(X)$ exists
  (so predicates can be instantiated at all sort-correct terms)
- no such pullback is an isomorphism
  (so no instance of a predicate is true *in the syntax*)

There is a construction that allows us to add generic predicates to a category, of which we will consider a simple special case.

A Category With a Generic Predicate ($\mathbb{C}[X]$)

For $\mathbb{C}$ a category with finite products and $b : \mathbb{C}$ (the sort at which we want to add a generic predicate), we construct the category $\mathbb{C}[X]$ to have:

objects pairs $\langle A, S \rangle$ where $A : \mathbb{C}$ and $S \subset \mathbb{C}(A \to b)$ is finite

arrows triples $\langle A, S \rangle \xrightarrow{f} \langle B, T \rangle$ where $f : \mathbb{C}(A \to B)$ such that

$$f \cdot T \subset S$$

we call f the *label* of this arrow

We claim that this category will embed an isomorphic copy of $\mathbb{C}$, and contain a new generic predicate of sort b.

Properties of $\mathbb{C}[X]$

This construction preserves the categorical properties we want our syntax to have:

- $\mathbb{C}[X]$ has finite products:
  - terminal object $\langle \mathbf{1}, \emptyset \rangle$
  - binary products $\langle A, S \rangle \times \langle B, T \rangle = \langle A \times B \ , \ \pi_{1_\mathbb{C}} \cdot S \cup \pi_{2_\mathbb{C}} \cdot T \rangle$
- $\mathbb{C}[X]$ inherits canonical structure from $\mathbb{C}$
- The **inclusion functor** $\iota : \mathbb{C} \to \mathbb{C}[X]$, mapping:

$$\begin{array}{rcl} A & \longmapsto & \langle A, \emptyset \rangle \\ f : A \to B & \longmapsto & \langle A, \emptyset \rangle \xrightarrow{f} \langle B, \emptyset \rangle \end{array}$$

is a continuous full and faithful embedding.

Special Subobjects

indeterminate subobject (X) In $\mathbb{C}[X]$, the *indeterminate subobject* X of sort b is the (monic representing its) subobject:

$$\langle b, \{\mathrm{id}_b\}\rangle \overset{id_b}{\rightarrowtail} \langle b, \emptyset\rangle$$

atomic goal the pullback of the X along the ι-image of an arrow $t : \mathbb{C}(A \to b)$:

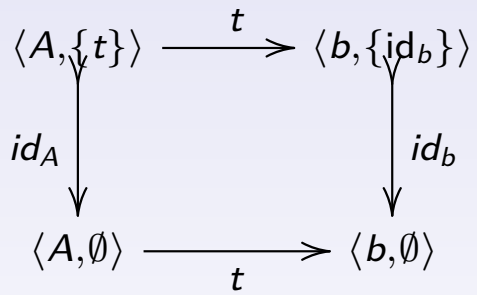
$$\langle A, \{t\}\rangle \overset{\mathrm{id}_A}{\rightarrowtail} \langle A, \emptyset\rangle$$

goal a monic labeled by the identity: $\langle A, S\rangle \overset{\mathrm{id}_A}{\rightarrowtail} \langle A, \emptyset\rangle$

The Indeterminate Subobject is Generic

proposition Indeterminate subobject X of sort b is a generic predicate with respect to the image of arrows of $\mathbb{C}$ under the inclusion map $\iota : (\mathbb{C} \rightarrow \mathbb{C}[X])$

proof For any $t : \mathbb{C}(A \rightarrow b)$



The left arrow is the pullback of X along $\iota(t)$.
It will never be an isomorphism since $t \notin \emptyset$.

Syntactic Categories of Generic Predicates

- We can either iterate or generalize this construction to add as many generic predicates (of whatever sorts) as we need for the relation symbols of a language.
- We will call such generic predicate categories $\mathbb{C}[\vec{X}]$, where $\vec{X}$ is a sequence of the predicate symbols we are interested in.

Interpretation of $\mathbb{C}[\vec{X}]$ in the Yoneda Topos

A continuous functor

$$[\![\_]\!] : \mathbb{C}[\vec{X}] \to \mathrm{SET}^{\mathbb{C}^{\mathrm{o}}}$$

- agreeing with the Yoneda embedding on $\iota(\mathbb{C})$
- mapping the source of X_i to some canonical subobject (i.e. pointwise subset) of $\mathbb{C}(\_ \to b_i)$
  (i.e. $[\![\_]\!](\Box X_i)(A) \subset \mathbb{C}(A \to b_i)$)

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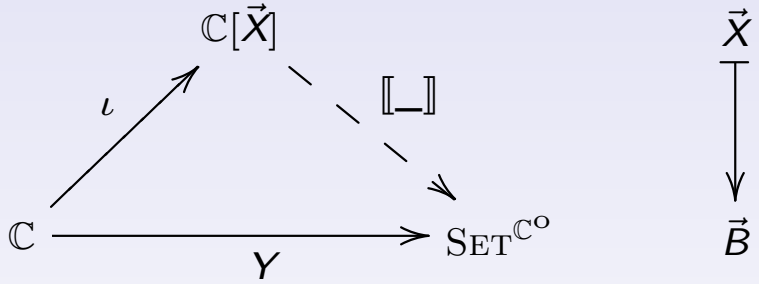


a universal mapping property tells us that this is enough to uniquely determine $[\![\_]\!]$

Category of Interpretations

- Let $\mathcal{I}^{\mathbb{C}}_{\vec{X}}$ be the set of interpretations of $\mathbb{C}[\vec{X}]$ in $\mathrm{SET}^{\mathbb{C}^{\mathrm{o}}}$.
- We can make $\mathcal{I}^{\mathbb{C}}_{\vec{X}}$ into a poset, $(\mathcal{I}^{\mathbb{C}}_{\vec{X}}, \sqsubset)$ where

$$[\![\_]\!] \sqsubset [\![\_]\!]' \iff \forall (X_i \in \vec{X}).\ [\![X_i]\!] \subset [\![X_i]\!]' \text{pointwise}$$

- This poset $(\mathcal{I}^{\mathbb{C}}_{\vec{X}}, \sqsubset)$ is a complete lattice, with meet and join given pointwise and top and bottom elements:

$$[\![X_i]\!]^{\top} = \mathbb{C}(\_, b_i) \xrightarrow{id} \mathbb{C}(\_, b_i)$$
$$[\![X_i]\!]^{\bot} = \mathbf{0} \xrightarrow{\mathrm{i}} \mathbb{C}(\_, b_i) \qquad (\mathbf{0} = \lambda x.\ \emptyset \text{ is initial})$$

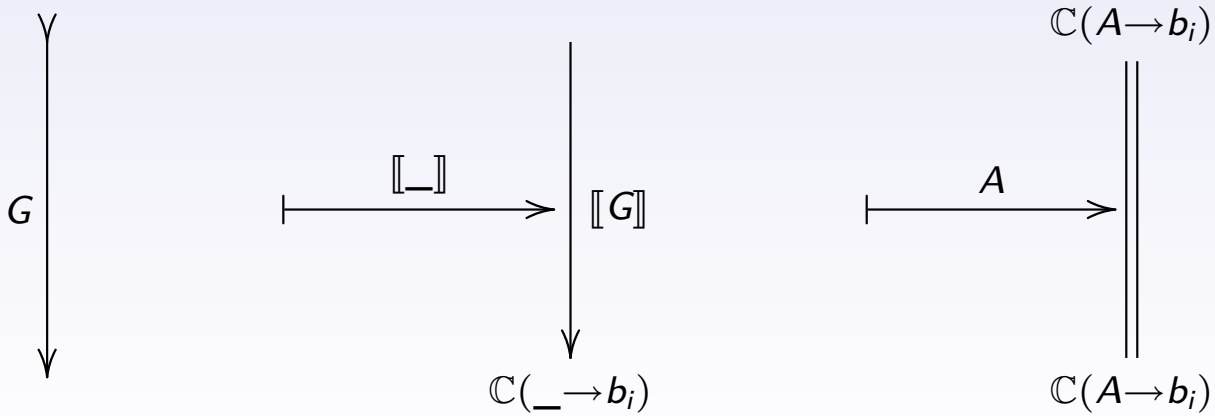
- Let $\mathrm{INT}^{\mathbb{C}}_{\vec{X}}$ be this poset, as a category.

Models and Truth

model An interpretation $[\![\_]\!] : \mathrm{INT}^{\mathbb{C}}_{\vec{X}}$ is a *model* of a definite clause program P if for every clause $(hd \leftarrow tl) \in P$, we have $[\![tl]\!] \subset [\![hd]\!]$, pointwise in $\mathrm{SET}^{\mathbb{C}^o}$.

truth of a goal in an interpretation A goal G is true in interpretation $[\![\_]\!]$ if its image is an isomorphism.

$\mathbb{C}[\vec{X}]$ $\quad$ $\mathrm{SET}^{\mathbb{C}^o}$ $\quad$ SET



Soundness

of SLD resolution with respect to models of a program

theorem

If interpretation $[\![\_]\!] : \mathbb{C}[\vec{X}] \to \mathrm{SET}^{\mathbb{C}^{\mathrm{o}}}$ is a model for program P and G is a goal, if there is an categorical SLD derivation $G \rightsquigarrow \overset{\Theta}{\cdots} \rightsquigarrow \top$ with computed answer substitution Θ, then $\Theta^{\#} G$ is true in $[\![\_]\!]$.

Immediate Consequences (E_P)

E_P is the categorical analog to T_P of Kowalski-van Emden semantics.

$$E_P : \mathrm{INT}^{\mathbb{C}}_{\vec{X}} \to \mathrm{INT}^{\mathbb{C}}_{\vec{X}}$$

- E_P is a continuous operator on $\mathcal{I}^{\mathbb{C}}_{\vec{X}}$ and thus has a least fixed point, $[\![\_]\!]^*$, which is the categorical analogue of the least Herbrand model.
- $[\![\_]\!]^*$ allows us to prove a completeness theorem for categorical SLD resolution as well.

5 beyond definite clause logic

Toward extensions

- so far, we have developed a categorical syntax, derivation system and bottom-up semantic only for definite clause logic, which, as we pointed out in the beginning lacks desirable features
- some of these can be added by changing the syntactic category
- some can be added by passing to indexed categories

Syntactic Categories with more Structure

Consider a simple predicate for the factorial relation

$$\texttt{fact}(0, 1).$$
$$\texttt{fact}(\texttt{X} + 1, (\texttt{X} + 1) * \texttt{Y}) : -\texttt{fact}(\texttt{X}, \texttt{Y}).$$

In a syntactic category in which fact is generic (and in PROLOG), this will not do what we want.
However, we could imagine this as a program for a syntactic category *modulo arithmetic*.
Then, for example, the term $2 + 4$ and the term $3 * 2$ would be *the same arrow in the syntax*.

(strict) Indexed Categories

indexed category For category $\mathbb{C}$, a functor $P : \mathbb{C}^{\mathrm{o}} \to \mathrm{CAT}$

For $\sigma : \mathbb{C}$, $P(\sigma)$ is called the *fiber over* σ
and for $f :: \mathbb{C}$, $P(f)$ is called a *reindexing functor*.

logic programming in indexed categories

- The base category $\mathbb{C}$ becomes the "state" we wish to index over. In the simplest case, this is the category of sorts and terms.
- Predicates are objects in the fibers over their state.
- The action of the reindexing functors and the structure of the fibers together with the program generate the resolution steps.

Hereditarily Harrolp Logic

- Hereditarily Harrop logic extends clausal logic by allowing implication and in goals:

$$G ::= \top \mid A \mid G \wedge G \mid G \vee G \mid D \rightarrow G \mid \exists_{x:\alpha} G \mid \forall_{x:\alpha} G$$
$$D ::= A \mid A \Leftarrow G \mid D \wedge D \mid \forall_{x:\alpha} D$$

- In [4] it was shown that this logic fragment can be made into a logic programming language with a proof search system called **uniform proof**, that for every non-atomic formula, applies the natural deduction proof rule that introduces its primary connective or quantifier.
- indexing over programs, allows us to handle implication in goals ([2])
- indexing over freely-adjoined constants gets us universal quantification in goals as well (Krishnan)

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